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ABSTRACT

This PME-NA proceedings volume contains the full text of 41 research papers. In addition, brief usually one-page reports, are provided for 11 oral presentations and 13 poster sessions. The full research reports are as follows: "Cognitive Analysis of Chinese Students' Mathematical Problem Solving" (J. Cai and E. A. Silver); (2) "Mathematical and Verbal Abilities in Mathematical Problem Solving by Talented Students" (S. D. Moore); (3) "One Student's Effort in Resolving His Self-Generated Measurement Problem: 'See If It Would Work With a Triangle.'" (A. Reynolds and G. H. Wheatley); (4) "Problem Solving Using Arithmetic and Algebraic Thinking" (A. R. Teppo and W. W. Esty); (5) "The Development of Children's Concept of Unit--Grades 4-8" (S. J. Lamon); (6) "Interaction and Fraction Knowledge: Children's Construction of the Iterative Partitioning Scheme" (R. Tzur); (7) "Using Case Studies to Promote Instructional Change" (B. Armstrong, And Others); (8) "The Validity of Concept Maps as a Research Tool in Remedial College Mathematics" (J. Laturno); (9) "Collaborative Action Research in Mathematics Education" (A. M. Raymond); (10) "Middle School Students' Perceptions of Their Everyday Mathematics Usage" (J. O. Masingila); (11) "A Basis for Equity in Mathematics Education: An Experiment In Cultural Course Development" (N. C. Presmeg); (12) "A Case of Equity Reform In Mathematics" (B. F. Risacher); (13) "The Negotiation of Social Norms in a Mathematics Class" (S. D. Trowell and G. H. Wheatley); (14) "Exploring the Social in Social Construction" (S. R. Williams and K. M. C. Ivey); (15) "Empowering Students to Talk About Mathematics In a Seventh-Grade Classroom" (V. M. Adams); (16) "Confidence in Answer Keys Among College Calculus Students" (H. T. Barton); (17) "Enhancing Student Interest in Mathematics Using Integrative Curricula" (M. Mitchell); (18) "The Effects of Psychosocial Variables on Middle School Student Problem-Solving Achievement in Mathematics" (J. L. Nath, And Others); (19) "Justifying the Reasonableness of Answers: Processes of Middle

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Proceedings of the

Sixteenth Annual Meeting

NORTH AMERICAN CHAPTER

of the

INTERNATIONAL GROUP

for the

PSYCHOLOGY OF MATHEMATICS EDUCATION

Volume 2:
Research Papers, Oral Reports, and Posters (continued)

PME-NA XVI

November 5-8, 1994

Louisiana State University

Baton Rouge, Louisiana, USA

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Baton Rouge LA USA

Editor:
David Kirshner

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HISTORY AND AIMS OF THE PME GROUP

PME came into existence at the Third International Congress on Mathematical Education (ICME 3) held in Karlsruhe, Germany, in 1976. It is affiliated with the International Commission for Mathematical Instruction.

The major goals of the Group and of the North American Chapter (PME-NA) are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
2. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers;
3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

Editor's Preface

The program for the 16th Annual Conference of PME-NA was established through the active participation of the Program Committee during the year leading up to the 1994 meeting. There were three invited session themes. *Reification in Secondary and K-12 Education* was selected to serve as a bridge between mathematicians and mathematics educators within the PME-NA family. This theme was addressed in a plenary session jointly presented by Anna Sfard, Hebrew University in Jerusalem, and Patrick Thompson, San Diego State University. *Situated Cognition Theory* was selected as a theme to explore philosophical and epistemological concerns that have been important to the PME and PME-NA community for a number of years. This theme was addressed in a second plenary session delivered by Tony Whitson, University of Delaware, with Judit Moschkovitch of the Institute for Research on Learning, in Palo Alto CA, serving as discussant. The final invited theme, *Rational Number Concepts*, was addressed in an invited discussion group organized by Thomas Kieren, University of Alberta. This discussion group used a new format in which the membership was invited to submit one page synopses of positions and views of rational number concepts to Dr. Kieren to be orchestrated into the discussion group session. These one page synopses are included in the Proceedings.

A special feature of the 1994 meeting were the *Technology Focus Groups* organized by James J. Kaput, University of Massachusetts, North Dartmouth. Each conference registrant was invited to participate in one of ten such groups in which developers of technology-based mathematics learning environments led investigations into problems in the psychology of mathematics education that arise in such learning environments. The groups met for a three or four hour block of time in a computer laboratory on the Louisiana State University campus. Our purpose in this session format was to broaden the base of understanding within the mathematics education community of the problems and possibilities of technological innovation. One page synopses of these groups are included.

The 67 research reports, 16 short oral reports, and 28 posters that constituted the main body of the conference program span a variety of issues and are organized topically within these *Proceedings*. This is a change from previous PME-NA proceedings in which only research reports were organized topically. Additionally one symposium and 5 discussion groups were presented, and synopses of these are included in the *Proceedings*. Proposals for all sessions were reviewed by three reviewers with expertise in the area of the submission. Cases of disagreement among the reviewers were resolved in a meeting of the Program Committee in New Orleans in April of 1993. This procedure resulted in the rejection or reassignment of about 20% of the proposals.

Thanks are due to all participants in PME-NA XVI, but especially to the Executive and Program Committees, to the reviewers, to the LSU University community for use of computer laboratories, to Dean Larry Pierce of the College of Education and to Dr. Neil Mathews, Chair of the Department of Curriculum & Instruction for their support of conference activities, to Kathy Carroll for technical coordination of the proceedings, to Doug Bourgeois and Brandy Baechle of LSU Conference Services for their tireless work, and to Clint Kaufmann for his valuable assistance.

David Kirshner
September, 1994

PME-NA 1994 REVIEWERS

Awtry, T
Bebout, H
Behr, M
Berenson, S
Bowman, A
Brenner, M
Brown, D
Bohlin, C
Bohlin, R
Bondoreur, P
Campbell S
Corwin, R
Chapin, S
Curcio, F
Davenport, L
Dawson, A
DeGuire, L
Ebert, C
Edwards, L
Fleener, J
Friedman, L
Ganter, S
Gaulin, C
Gordon, A
Grover, B
Goodson-Espy, T
Golding, T
Graham, K
Gorowara, C
Harel, G
Harik, F
Harnisch, D
Ham, J
Hart, L
Heid, M
Jakubowski, E
Kaufmann, C
Kirshner, D
Kloosterman, P
Koch, L
Krussel, C
Kaput, J
Lamb, C
Lambdin, D
Lamon, S
Lester, F
Maher, C
Masingila, J
Massey, J
Mayer, R
McCoy, L
McNeal, B
Mitchell, M
Meredith, J
Magidson, S
Miller, D
Mukhopadhyam,
O'Shea, T
O'Callaghan, B
Pence, B
Post, T
Presmeg, N
Preston, R
Raymond, A
Romberg, T
Reynolds, A
Risacher, B
Schifter, D
Selden, A
Selden, J
Senger, B
Shealy, B
Smith, E
Sowder, J
Storygard, J
Simon, M
Swafford, J
Schroeder, T
Underhill, B
Watanabe, T
Wheatley, G
Walen, S
Wilson, M
Wandersee, J
Yackel, E
Zazkis, R

Table of Contents

Contents of Volume I

PREFACE

PLENARY SESSIONS

Problems Of Reification: Representations And Mathematical Objects 1-3
Anna Sfard and Patrick W. Thompson

Elements of a Semiotic Framework for Understanding Situated and Conceptual Learning 1-35
James Anthony Whitson

INVITED DISCUSSION GROUP

Reflections And Interactions On Rational Number Thinking, Learning And Teaching: An Introduction To The Discussion Group 1-53
Thomas E. Kieren

Imagination and Mathematics: The Size of a Ratio 1-57
Susan J. Lamon

Classroom Activities to Prepare Early Elementary Students for Proportional Reasoning 1-58
Susan J. Lamon

Ratio Sense 1-59
Susan J. Lamon

A Comparison of Two Approaches to Develop Partitioning Skills As Precursors to Learning Basic Fractions 1-60
Sylvia R. Taube

Children's Ideas About Partition and Sharing 1-61
Olimpia Figueras

TECHNOLOGY FOCUS GROUPS

Visualizing Aspects Of Number Theory Through Spreadsheet Modeling 1-65
Sergei Abramovich and Azita Manouchehri

Linking The Particular And The General In Algebra 1-66
Daniel Chazan, Judah Schwartz, and Michal Yerushalmy

Geometry Technology Focus Group 1-67
Douglas H. Clements, Julie Sarama Meredith, and Michael T. Battista

Tools For Knowledge Construction: Toolbook And SPPPC In Classroom Settings 1-68
Michael L. Connell and Delwyn L. Harnisch

LOGO in the Classroom: Recognizing Socio-cultural Contexts in Technology-based Learning Environments 1-69
Paula K. Hooper and Erick Smith

Pacesetter: Mathematical Modeling Curriculum 1-70
Perry Lanier, Melissa Dennis, Robert Guth, and William Fisher

Children's Construction Of Fractions Using Tools For Interactive Mathematical Activity (TIMA) Microworlds 1-71
John Olive

Teaching And Learning Geometry With The Aid Of Dynamic Construction Tools Such As The Geometer's Sketchpad 1-72
John Olive

Jumping And Running: Using Technology To Capture The Real World 1-73
Andee Rubin and Nathan Kimball

The Adventures Of Jasper Woodbury And The Adventuremaker: Technologies That Support Mathematical Problem Solving And Problem Posing For Realistically Complex Problems 1-74
Susan Williams, Mitchell J. Nathan, Joyce L. Moore, Sashank Varma, Susan R. Goldman, and The Cognition and Technology Group at Vanderbilt

SYMPOSIUM AND DISCUSSION GROUPS

Sympos.: Looking At Assessment In Mathematics: Building A Framework 1-77
Kathy Kelly-Benjamin, Paula S. Krist, Marie Revak, and Carol Kehr Tittle

Discussion Group: Emerging Standards For Research 1-81
Thomas J. Cooney and Frank K. Lester, Jr.

Mathematics Classrooms as Complex Adaptive Systems 1-82
A. J. (Sandy) Dawson

Developing Training and Research Paradigms for the Preparation of Elementary Mathematics and Science Lead Teachers 1-83
Susan N. Friel, Rebecca B. Corwin, and Robert G. Underhill

Conceptual Understanding Of Common And Decimal Fractions 1-84
Douglas Owens

Interactive Multimedia: Its Impact On Research 1-85
Teri Perl, Carolyn Maher, and Anne Teppo

Children's Mathematical Knowledge And Construction Of Units 1-86
Tad Watanabe

RESEARCH PAPERS, ORAL REPORTS, AND POSTERS

Advanced Mathematical Thinking

RESEARCH PAPERS

Factors In Learning Linear Algebra 1-89
Guershon Harel

Articulations Between The Settings, Numeric, Algebraic And Graphic Related To The Differential Equations 1-95
Arturo Hernandez and Fernando Hitt

Image Structures And Reification In Advanced Mathematical Thinking: The Concept Of Basis 1-102
Libby Krussel

A Survey Of Tertiary Students' Entry Level Understanding Of Mathematics Vocabulary 1-109
L. Diane Miller and Brian White

Constructing The Derivative in First Semester Calculus
Bob Speiser and Chuck Walter 1-116

ORAL REPORTS

Using A Computer Laboratory Setting (CLS) To Teach College Calculus 1-123
Nkechi Madonna Agwu

Constructivist Approach For Helping Students Link The Polar Plane To The Cartesian Plane With A Graphing Calculator 1-124
Yvelyne Germain-McCarthy

Student Understanding Of Derivative In A Technology Enhanced Classroom 1-125
Michelle J. Zandieh

POSTERS

An Empirical Investigation Of Graphing Calculator Use And Guided Discovery Style Teaching In Differential Calculus 1-126
George L. Emese

Calculus For All: Simulations For Calculus Learning In Middle School 1-127
James J. Kaput

Informal Approaches to Limits Through Topics in K-12 Mathematics Curriculum 1-128
Brenda Lee

Algebraic and Quantitative Thinking

RESEARCH PAPERS

Visual Salience In Algebraic Transformations 1-131
Thomas Awtry and David Kirshner

Preparing Students For Algebra: The Role Of Multiple Representations In Problem Solving 1-138
Mary E. Brenner and Bryan Moseley

Introducing Algebra With Programmable Calculators 1-145
Tenoch Cedillo Avalos

Blind Calculators', 'Denotation' Of Algebra Symbolic Expressions, And 'Write False' Interviews 1-152
Jean-Philippe Drouhard, Francois Leonard, Maryse Maurel, Michele Pecal, and Catherine Sackur

School Algebra: Syntactic Difficulties In The Operativity With Negative Numbers 1-159
Aurora Gallardo and Teresa Rojano

A Constructivist Explanation Of The Transition From Arithmetic To Algebra: Problem Solving In The Context Of Linear Inequality 1-166
Tracy Goodson-Espy

Multi-Tasking Algebra Representation 1-173
Leah P. McCoy

ORAL REPORTS

Quantitative Mental Models Of Linear Functions 1-180
Joyce L. Moore

Climbing The Incline Of The Slope Concept 1-181
Vicki Schell

POSTERS

The Distributive Law In Arithmetic And Algebra And The Cognitive Processes Of College Level Remedial Students 1-182
Bronislaw Czamocha

Fourth Graders Invent Ways Of Computing Averages 1-183
Constance Kamii

See & Say: An Analysis Of Verbal And Visual Representation In First Year Algebra Students 1-184
Mary Katherine Newberg

13

Misconceptions Of Variable 1-185
Ricardo Quintero, Araceli Reyes, Maria Trigueros, and Sonia Ursini

A.C.T. In Algebra 1-186
Lara S. Schmidt and Thomas R. Swanson

Students Interpretation Of Variation Problems In Graphical Setting 1-187
Maria Trigueros

Assessment

RESEARCH PAPERS

Assessing Student Responses To Performance Assessment Tasks 1-191
Susan Hillman

Multi-faceted Inferences From An Interview Assessment 1-199
Thomas L. Schroeder

POSTERS

Assessing Students' Mathematical Activity In The Context Of Design Projects 1-206
Judit N. Moschkovich

A Model For Assessing Children's Mathematical Thinking In Classrooms Where Reform Is Taking Place 1-207
Roberta Schorr, Carolyn Maher, and Alice Alston

Cognitive Modalities

RESEARCH PAPERS

Visualization In Mathematics: Spatial Reasoning Skill And Gender Differences 1-211
Lynn Friedman

Negative Consequences Of Rote Instruction For Meaningful Learning 1-218
Dolores Simoneaux and David Kirshner

Long Term Relationship Between Spatial Ability And Mathematical Knowledge 1-225
Grayson H. Wheatley, Dawn L. Brown, and Alejandro Solano

ORAL REPORTS

The Role Of Individual Reconstructions Of Mathematical Concepts In Dyadic Problem-Solving 1-232
Draga Vidakovic

POSTERS

A Category-Theoretic Description Of The Mechanisms Underlying Shifts Of Cognitive Stage And Shifts In Levels Of Geometric Thought Development 1-233
Livia P. Denis and Jose Reyes

Lego-Logo, Spatial Reasoning And Girls 1-234
Laurie D. Edwards

Turtle Math: A Logo Environment Grounded In Research 1-235
Julie Sarama Meredith and Douglas H. Clements

Epistemology

RESEARCH PAPERS

The Role Of Context In Mathematical Activity 1-239
David Clarke and Sue Helme

The Relationship Between Preservice Teachers' Metaphors For Mathematics Learning And Habermasian Interests 1-247
M. Jayne Fleener and Anne Reynolds

POSTERS

When A Solution To A Mathematical Problem Becomes An Acceptable Proof 1-254
Ewa Prus-Wisniowska

Function Concepts

RESEARCH PAPERS

An Investigation Into The Development Of Student Understanding Of The Graphs Of Polynomial Functions Of Degree Greater Than Two: Results And Implications 1-257
Judith E. Curran

A Modeling Approach To Constructing Trigonometric Functions 1-264
Helen M. Doerr

Students' Conceptions Of Functions In A Computer-Rich Problem Solving Environment 1-271
Brian R. O'Callaghan and David Kirshner

Towards An Algebraic Notion Of Function: The Role Of Spreadsheets 1-278
Teresa Rojano and Rosamund Sutherland

POSTERS

Student Conceptions Of Graphical Models Of Linear And Non-Linear Functions 1-285
Wendy N. Coulombe

Geometric Thinking

RESEARCH PAPERS

Students' Development Of Length Measurement Concepts Using A Specially-Designed Turtle Graphics Environment 1-289
Douglas H. Clements, Michael T. Battista, Julie Sarama Meredith, Sudha Swaminathan, and Sue McMillen

The Role Of Language In Geometric Concept Formation: An Exploratory Study With Deaf Students 1-296
Marguerite M. Mason

ORAL REPORTS

Geometry And Visualization In Different Environments 1-303
Marcela Santillan Nieto

Probability and Statistics

RESEARCH PAPERS

Development Of The Concept Of Randomness 1-307
Kathleen E. Metz

POSTERS

Paths to Interpreting Statistical Information 1-314
Andrew Ahlgren

LIST OF AUTHORS 1-315

Contents of Volume II

RESEARCH PAPERS, ORAL REPORTS, AND POSTERS (continued)

Problem Solving

RESEARCH PAPERS

Cognitive Analysis Of Chinese Students' Mathematical Problem Solving: An Exploratory Study 2-3
Jinfa Cai and Edward A. Silver

Mathematical And Verbal Abilities In Mathematical Problem Solving By Talented Students 2-10
Sara Delano Moore

One Student's Effort In Resolving His Self-Generated Measurement Problem: "See If It Would Work With A Triangle." 2-17
Anne Reynolds and Grayson H. Wheatley

Problem Solving Using Arithmetic And Algebraic Thinking 2-24
Anne R. Teppo and Warren W. Esty

ORAL REPORTS

Solving Arithmetic Word Problems: Cross-National Comparisons 2-31
Irene T. Miura and Yukari Okamoto

Rational Number Concepts

RESEARCH PAPERS

The Development Of Children's Concept Of Unit--Grades 4-8 2-35
Susan J. Lamon

Interaction And Fraction Knowledge: Children's Construction Of The Iterative Partitioning Scheme 2-42
Ron Tzur

ORAL REPORTS

Partitioning And Equivalence In A Geometric Setting 2-49
Vicki Schell and Marguerite Mason

POSTERS

Developing Conceptual Grouping Structures: New Software Approaches 2-50
James Kaput

Research Methods

RESEARCH PAPERS

Using Case Studies To Promote Instructional Change 2-53
Barbara Armstrong, Judith Sowder, and Randolph Philipp

The Validity Of Concept Maps As A Research Tool In Remedial College Mathematics 2-60
Jan Laturno

Collaborative Action Research In Mathematics Education: A Tale Of Two Teacher-Researchers 2-67
Anne M. Raymond

Social and Cultural Factors

RESEARCH PAPERS

Middle School Students' Perceptions Of Their Everyday Mathematics Usage 2-77
Joanna O. Masingila

A Basis For Equity In Mathematics Education:
An Experiment In Cultural Course Development 2-84
Norma C. Presmeg

A Case Of Equity Reform In Mathematics:
The Students And Their Response 2-91
B. F. Risacher

The Negotiation Of Social Norms In A Mathematics Class 2-98
Sandra Davis Trowell and Grayson H. Wheatley

Exploring The Social In Social Construction 2-105
Steven R. Williams and Kathy M. C. Ivey

Students' Attitudes and Beliefs

RESEARCH PAPERS

Empowering Students To Talk About Mathematics
In A Seventh-Grade Classroom 2-115
Verna M. Adams

Confidence In Answer Keys Among College Calculus Students:
Finding Out The Truth About Santa Claus 2-122
Hylie T. Barton

Enhancing Student Interest In Mathematics Using Integrative Curricula 2-129
Mathew Mitchell

The Effects Of Psychosocial Variables On Middle School
Student Problem-Solving Achievement In Mathematics 2-136
Janice L. Nath, Susan E. Williams, and Hersholt C. Waxman

Justifying The Reasonableness of Answers: Processes Of
Middle School Mathematics Students 2-143
Susan E. Williams and Juanita V. Copley

ORAL REPORTS

Voicing The Voiceless: A Mathematics Teacher's And A Student's
Views And Beliefs About Teaching And Learning Mathematics 2-150
Evangelina Diaz-Obando and Elizabeth M. Jakubowski

Student Authority In The Classroom 2-151
Laura Coffin Koch

POSTERS

When Do The Equilateral Triangles And The Congruency
Get To Be Boring? Introduce To Deltahedral Models 2-152
Eugenio Diaz Barriga Arceo

Teachers' Beliefs and Attitudes

RESEARCH PAPERS

A Teacher's Perception Of Time In A Mathematics Classroom: A Case Study Of Reflection On Reflection-In-Action 2-155
Kathy M. C. Ivey

Preservice Secondary Teachers' Beliefs About Mathematics And Their Expectations About Student Performance On Open-Ended Assessment Tasks 2-162
Barbara W. Grover and Patricia Ann Kenney

Changing The Mathematics Learning Environment In Relation To Beliefs, Knowledge, And Practices 2-169
Patricia A. Jaberg and Cheryl A. Lubinski

Connecting Orientation Towards Authority To First-Year Teachers' Thinking About Teaching 2-176
Barry E. Shealy

ORAL REPORTS

A First Year Secondary Mathematics Teacher's Orientation Toward Authority: Implications For Professional Development 2-183
Bridget Arvold and Thomas J. Cooney

POSTERS

Emerging And Evolving Themes: Case Studies Of Three Preservice Secondary Mathematics Teachers' Beliefs 2-184
Roger Day

Warning-Hard Hat Area: Falling Beliefs 2-185
Annette Ricks Leitze

Teacher Change

RESEARCH PAPERS

Transforming Mathematics Teaching In Grades K-8: The Role Of Material Resources In Supporting Teacher Change 2-189
Linda Ruiz Davenport

Affective Issues In Developing Mathematics Teaching Practice 2-196
Lynn T. Goldsmith and Linda Ruiz Davenport

Case Studies On Empowering Secondary Mathematics Teachers In Computer-Intensive Environments 2-203
M. Kathleen Heid, Rose Mary Zbiek, and Glendon Blume

Perceived Deficits In Middle Grades Mathematics Teaching 2-210
M. Kathleen Heid, and Susan J. Feeley

How Middle School Teachers Adjust To Change: Classroom Testing Of Materials From An Innovative Curriculum Project 2-217
Diana V. Lambdin and Ronald V. Preston

The Needs of Second Career, Secondary Mathematics Teachers: How Well Are They Met By Academic Programs And Inductive Processes? 2-224
Sheryl A. Maxwell

Development Of Classroom Social Norms And Mathematical Practices With Preservice Teachers 2-231
Betsy McNeal and Martin A. Simon

Teaching In An Era Of Reform: Mathematics In Elementary Classrooms 2-238
Paola Sztajn and Frank K. Lester, Jr.

Changing Practice: Describing Mathematics Teachers' Development Through A Modification Of Perry's Scheme 2-245
Melvin R. Wilson and Michael P. Goldenberg

ORAL REPORTS

Teacher Paradigm: Form And Transition 2-252
Sharon B. Walen

POSTERS

Extending K-8 Teacher Development In Mathematics To The Secondary Level: A Model For District Systemic Reform 2-253
Emily Dann, Roberta Schorr, and Alice Alston

Mississippi Mathematics Improvement in Secondary Schools Project (MISS) 2-254
Barbara Dougherty and J. Larry Tyler

A Reflective Cycle: The Evolution Of A Model Of Teacher Change 2-255
Thomas G. Edwards

Teacher Inquiry Groups: Collaborative Explorations Of Changing Practice 2-256
Jim Hammerman

Negotiating A Technological Shift: Teacher Perception Of The Implementation Of Graphing Calculators In The Classroom 2-257
Marcia L. Tharp, Robin L. Brown, and James A. FitzSimmons

20

Teachers' Conceptions of Mathematics

RESEARCH PAPERS

Teachers' Graphical Representations Of Rate Of Change 2-261
Sarah B. Berenson and Glenda S. Carter

Distributive Flaws: Latent Conceptual Gaps In Preservice Teachers' Understanding Of The Property Relating Multiplication To Addition 2-268
Stephen Campbell and Rina Zazkis

Elementary Teachers' Understanding And Implementation Of Unitizing Operations 2-275
Tena Long Golding and Merlyn J. Behr

Elementary School Teachers' Perceptions Of Algebra: The Role Of Modelling And Technology 2-282
Barbara J. Pence

Learning Mathematics While Teaching 2-289
Susan Jo Russell, Deborah Schifter, Virginia Bastable, Lisa Yaffee, Jill B. Lester, and Sophia Cohen

Teachers' Changing Conceptions Of The Nature Of Mathematics: A Developmental Model 2-296
Deborah Schifter

The Fundamental Theorem Of Arithmetic: Used and Confused 2-303
Rina Zazkis and Stephen Campbell

ORAL REPORTS

Calculus Teachers' Recognition--In Different Contexts--Of The Concept Of Derivative 2-310
Rodrigo Cambray-Nunez

An Investigation of Preservice Teachers' Understanding of Rational Numbers 2-311
Christine Carrino Gorowara

Prospective Secondary Mathematics Teachers' Subject Matter Knowledge: The Concept of Limit As An Example 2-312
Brenda Lee

Hands-on Assessment of Elementary Teachers' Understanding of Whole Number, Fraction, and Decimal Concepts, Principles, and Processes 2-313
Olga M. Ramirez, John E. Bernard, and Arturo Olivarez

POSTERS
Preservice Teachers' Perceptions Of Procedural And Conceptual Problems 2-314
Kathryn Skipper Reinke

Teachers' Understanding of Students' Understanding

RESEARCH PAPERS
Delineating The Transformation Of Subject-Matter Knowledge Into Pedagogical Content Knowledge 2-317
Christine L. Ebert

Pedagogical Content Knowledge, Curricular Knowledge And Teacher Change 2-324
Beverly S. Rich, Cheryl A. Lubinski, and Albert D. Otto

POSTERS
Integrated Media In Preservice Teacher Education To Explore Children's Mathematical Learning 2-331
Linda Barron and Elizabeth Goldman

Traditional And Non-Traditional Pre-Service Mathematics: An Investigation Of Factors That Influence The Acquisition Of Pedagogical Content Knowledge 2-332
Christine L. Ebert and Billie F. Risacher

Pathfinder Analysis: Using Technology To Map The Mathematical Pedagogical Content Knowledge Of Preservice Elementary Teachers 2-333
Oakley D. Hadfield

LIST OF AUTHORS 2-335

Problem Solving

RESEARCH PAPERS

Cognitive Analysis Of Chinese Students' Mathematical
Problem Solving: An Exploratory Study 3
Jinfa Cai and Edward A. Silver

Mathematical And Verbal Abilities In Mathematical
Problem Solving By Talented Students 10
Sara Delano Moore

One Student's Effort In Resolving His Self-Generated Measurement
Problem: "See If It Would Work With A Triangle." 17
Anne Reynolds and Grayson H. Wheatley

Problem Solving Using Arithmetic And Algebraic Thinking 24
Anne R. Teppo and Warren W. Esty

ORAL REPORTS

Solving Arithmetic Word Problems: Cross-National Comparisons 31
Irene T. Miura and Yukari Okamoto

COGNITIVE ANALYSIS OF CHINESE STUDENTS' MATHEMATICAL PROBLEM SOLVING: AN EXPLORATORY STUDY

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The computational performance and problem-solving processes of 93 Chinese sixth-grade students were examined on four routine computation exercises and two nontrivial problems. Chinese students had little difficulty with computation, whether decontextualized or embedded within a story problem. Nevertheless, analysis of students' solutions revealed that the problem solving of Chinese students is similar in many ways to that reported in other analyses of American students. In particular, Chinese students made similar interpretation errors to those reported for American students when solving a division-with-remainder problem, and they used similar strategies and made similar errors as those reported for American students when solving a problem involving visual patterns with underlying mathematical regularities.

During the past several decades, a number of cross-national studies have been conducted either to examine differences in mathematical proficiency among students from various countries or to understand the possible contribution of various factors to the observed differences (e.g., Husen, 1964; Lapointe et al., 1992; Robitaille & Garden, 1989; Stevenson et al., 1990). Since China did not participate in either the First International Mathematical Study (FIMS) or the Second International Mathematical Study (SIMS), somewhat less is known about how Chinese students compare internationally. Nevertheless, a few cross-national comparative studies (e.g., Lapointe et al., 1992; Stevenson et al., 1990) have included Chinese students, and these studies have found that Chinese students have a high level of mathematical proficiency. For example, Stevenson et al. (1990) compared the achievement of American first-grade and fifth-grade students (in Chicago) on nine mathematical topic areas with a similar group of Chinese students (in Beijing). Stevenson et al., (1990) reported that Chinese students outperformed American students on almost every mathematical topic. This cross-national study, like the others involving Chinese students, used multiple-choice format tasks to measure students' mathematical performance; a qualitative, cognitive analysis of Chinese students' mathematical problem-solving processes has not been undertaken.

Since China has a strong reputation for developing students' mathematical problem solving (Stanic & Kilpatrick, 1989), and since little is known in the U. S. about Chinese students' mathematical problem-solving processes, it seemed fruitful to undertake an exploratory examination of Chinese students' mathematical performance on a set of computation tasks and pair of mathematical problems.

METHOD

Subjects

A total of 93 Chinese sixth-graders in two provinces participated in the study. The students who participated were judged to be of average ability by their teachers.

Tasks

Each student received a booklet containing four computational exercises, a division-with-remainder (DWR) story problem, and a pattern problem (See Appendix A). The booklet also contained a mathematical problem-posing task, in which students were asked to pose mathematical problems based on a given situation, but the results for this task are not reported here. Students had 30 minutes to complete written responses for all tasks. The tasks were selected to reveal a range of students' mathematical performance, especially their thinking and reasoning processes, such as those connected with generalization and the simultaneous coordination of two dimensions in the pattern problem and the interpretation of a solution in the DWR problem. These problems were also convenient to use here because they have been used in studies with American students (Cai & Magone, 1993; Silver et al., 1993), thereby affording an opportunity to examine similarities and differences between Chinese and American students' thinking about the problems.

Data Coding

Each response for the computation exercises was coded as correct or incorrect. Student responses for the DWR problem were coded using a classification scheme similar to that used by Silver et al. (1993) to code U.S. students' responses to a similar DWR problem. Each response was examined with respect to four distinct aspects: (1) solution processes, (2) execution of procedures, (3) numerical answers, and (4) explanations of the solutions. Student responses for the pattern problem were coded using a classification scheme similar to that used by Cai and Magone (1993) to code U.S. students' responses to a similar problem. Each response was examined with respect to three distinct aspects: (1) correctness of the figures and evidence of a description, (2) solution strategies, and (3) drawing errors.

RESULTS

Results for the Computation Exercises

Students were quite successful in solving the computation exercises. In fact, for three of the exercises, more than 90% of the students had correct answers. A majority of the students (70%) had correct answers for all four computational exercises, although a substantial number of the students (30%) made an error on at least one exercise. The results for each problem are shown in the table below.

Problems	1/5 + 2/3	3480 ÷ 60	15.3 - 8.8	11.28 ÷ 3.6
% Correct	96	92	96	81

Results for the DWR Problem

Students recognized this as problem involving division, and they were able to perform well the associated computation. In fact, over 90% the students used division, and 92% of the students who did so were able to execute the procedure flawlessly. With respect to the numerical answers, nearly 30% of the students provided 32 as the answer; 40% of the students gave 31 as the answer; and another 20% of the students gave an answer that expressed the remainder as a fraction, as a whole number, or as a decimal.

Students provided various interpretations of their computational results. About 20% of the students provided appropriate and complete explanations to support their answers. For example, appropriate interpretations for the answer of 32 consisted of written statements, such as the following: "If you have 31 buses, there are 12 people left over; these 12 people cannot walk and they have to go; therefore, you need one more bus". Other appropriate explanations were sometimes used to support a final answer of 31. For example, one student wrote, "If you have 31 buses, there are 12 people left over. These 12 people can squeeze into the 31 buses, so they just need 31 buses." Interestingly, two students explained appropriately and completely that the answer could be either 31 or 32.

Some students failed to provide an explanations of the solution or provided only a procedural explanation. For example, about one-sixth of the students (16%) directly rounded up the remainder (.3, 1/3, or 12 persons) and provided the answer 32, but these students did not justify why they did so. It is clear that these students did not simply apply a rule for rounding numbers because the remainders were less than one-half and rule would not due to round up. Possibly these students realized that they needed to account for the remainder in some way, but they did not know how to explain it. Interestingly, another one-fourth of the students directly rounded down the remainder and provided an answer of 31, but these students also did not justify their approach. Only about 10% of the students explained their solution by mentioning that they applied a rounding rule.

Results For the Pattern Problem

More than 80% of the students drew both figures correctly and provided descriptions of how they obtained the correct figures. Not all responses including both the fifth and seventh figures also included a description of the method used, but if a description was given, the fifth and seventh figures were also given, and if the description was correct, then so were the figures.

Almost all students used one of six strategies, which presented and described in the Table below, to discover the regularities underlying the pattern in this task. In fact, 85% of the students used one of these six strategies; and 15% of the students provided descriptions that were so vague or incomplete that no strategy could be inferred. The first five strategies were more commonly used than the sixth strategy, and each of the first five was used with almost the same frequency (as noted in the Table). Appendix B shows two sample student responses.

Strategy **(14%)**	1:Students focused on the number of dots in the three rows of each figure as a triplet and induced the rule of the triplets (Fig. 1: [1, 2, 3]; Fig. 2: [2, 3, 4]; Fig. 3: [3, 4, 5]; etc.)
Strategy **(19%)**	2: Students looked at the dots in each row across the figures as a sequence and found the rule of each sequence (Row 1: {1,2,3,4, ···}; Row 2: {2,3,4,5, ···}; Row 3: {3,4,5,6, ···}).
Strategy **(18%)**	3: Students looked at the figures diagonally and realized each successive figure has one more diagonal of 3 dots. By adding one more diagonal column to the fourth figure, then students would get the fifth figure.
Strategy **(14%)**	4: Students realized that, from figure to figure, each row has one more dot than the corresponding row in the previous figure. From the fourth figure, the student added one dot to each row individually to get the fifth figure.
Strategy **(18%)**	5: Students focused on the total number of dots in each figure to describe how to get the next figure. The first figure has 6 circles, the second has 9, the third has 12, the fourth has 15, ···, therefore the seventh figure has 24 circles. In this case, the student did not mention the shape of each figure explicitly, and may actually have ignored the shape of the figures.
Strategy **(2%)**	6: Students removed the first row of the previous figure, then added a new bottom row that has one more dot than the previous bottom row, this would be the next figure.

A number of students made drawing errors. In fact, 19% of the students failed to draw either the fifth or seventh figure correctly. In order to find the correct fifth and seventh figures, students must simultaneously coordinate regularities along two dimensions in the set of figures: the number of dots and the shape. The majority of drawing errors resulted from the fact that students failed to coordinate along both dimensions. For example, some students drew all 18 dots in the fifth figure and 24 dots in the seventh figure, but did not produce the correct shapes of the figures (see response 2 in Appendix B). For those students who used the fifth strategy, more than half of them drew the fifth and seventh figures correctly in terms of the total number of dots, but did not maintain the correct shapes of the figures. Some other students drew the correct shape for the fifth figure, but they failed to produce the correct number of dots.

DISCUSSION

In this study Chinese students had very little difficulty with computation, whether it was decontextualized or, in the case of division computation, embedded within a story problem to be solved. This result is consistent with reports of Chinese students' computational prowess obtained from cross-national studies (e.g., Lapointe et al., 1992; Stevenson et al., 1990).

Although almost all Chinese students successfully completed a division computation in attempting to solve the DWR problem in this study, only a small proportion of the students were successful in obtaining and explaining a reasonable solution to the given problem. Chinese students' difficulties in solving the DWR problem, like those reported for American students on a similar problem (Silver et al., 1993), apparently derives not from the computation itself but from the sense-making requirement involved in interpreting the computational result. Thus, Chinese students had similar difficulties in solving the DWR problem as American students do.

Other similarities were noted between the problem solving for Chinese students and that reported elsewhere for American students. In particular, the types of strategies employed and the types of drawing errors made by Chinese students in this study in solving the pattern problem were similar to those reported for American students (Cai & Magone , 1993). For example, the six solution strategies used by Chinese students in this study were also reportedly used by

American students (Cai & Magone , 1993). And the types of drawing errors made by Chinese students, due to difficulties in coordinating multiple dimensions of the task, were also similar.

The analysis of both computation and nonroutine problem solving in this study affords a somewhat broader look at mathematical proficiency than has been undertaken in prior examinations of Chinese students. Moreover, the analysis of cognitive aspects of students' mathematical problem- solving processes affords a more complete picture of the similarities and differences between students in China and the U.S. In fact, this study suggests the feasibility of conducting rigorous cognitive analysis to capture students' in-depth mathematical thinking and reasoning, and this suggests a fruitful direction for future cross-national studies in mathematics (Cai, 1994).

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Appendix A: Tasks

Computation Exercises

$\frac{1}{5} + \frac{2}{3} = ?$　　$3480 \div 60 = ?$　　$15.3 - 8.8 = ?$　　$11.28 \div 3.6 = ?$

DWR Problem

(English version)

Students and teachers at Gunming elementary school will go by bus to have Spring sightseeing. There is a total of 1128 students and teachers. Each bus holds 36 people. How many buses are needed?

Answer: ____________________

Show all your solution processes below.

(Chinese version)

光明小学师生共 1128 人乘车去春游。
如果每辆车上能坐 36 人，
他们共需要多少辆车？

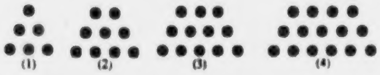
答案：___

写出你的全部解答过程。

Pattern Problem

(English version)

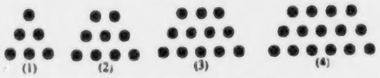
Look at the pattern below.



A. Draw the fifth figure.
B. Draw the seventh figure.
C. Describe how you knew what the 7th figure would look like.

(Chinese version)

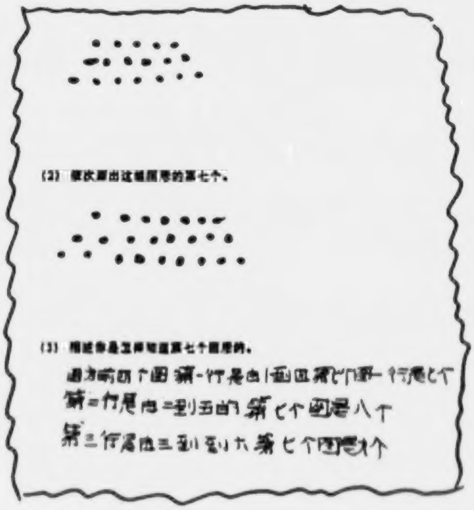
有一组图形按某种规律变化。



(1) 依次画出这组图形的第五个。
(2) 依次画出这组图形的第七个。
(3) 描述你是怎样知道第七个图形的。

Appendix B: Sample Student Responses

Response 1: Strategy 2

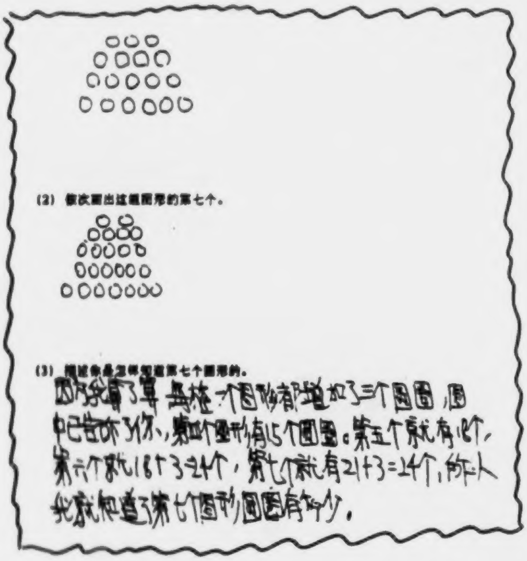


(2) 依次画出这组图形的第七个。

(3) 描述你是怎样知道第七个图形的。

因为前四个图 第一行是由1到四 第七个图第一行是七个
第二行是由二到五的 第七个图是八个
第三行是由三到到六 第七个图是九个

Response 2: Strategy 5



(2) 依次画出这组图形的第七个。

(3) 描述你是怎样知道第七个图形的。

因为我算了算 每一个图形都增加了三个圆圈，图中已经有了3个、第四个图形有15个圆圈。第五个就有18个，第六个就18+3=21个，第七个就有21+3=24个，所以我就知道了第七个图形圆圈有多少。

MATHEMATICAL AND VERBAL ABILITIES IN MATHEMATICAL PROBLEM SOLVING BY TALENTED STUDENTS

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Research suggests that talented students are likely to be better problem solvers and to use more metacognitive skills in problem solving. This paper presents research which identifies mathematical and verbal abilities separately to further the investigation of problem solving in talented students. Highly verbally able students were divided into two groups, of above average and exceptional mathematical ability, and were asked to complete a series of problem solving tasks. Once differences in computational skill were taken into account, the students performed equally well under quantitative analysis. Still, response analysis showed that the more mathematically able students were more likely to use the metacognitive skills traditionally associated with excellent problem solving.

BACKGROUND

Three common themes run through the literature about problem solving and talented students. First is the possible interplay between verbal and mathematical ability. The research of Benbow and Minor (1990) supports the separateness of these two gifted constructs. The work of Hunt and his colleagues (1975), as well as that of Leinhardt [1985; cited in Span (1986)], support the theory that these two abilities interact in successful problem solvers. The various definitions of giftedness used in studies of mathematical problem solving do not allow us to sort out the effect of mathematical ability when students are of similar verbal ability.

The second theme is the role of metacognition. Successful problem solvers focus on understanding the problem rather than on the numbers contained in the problem (Garofalo, 1992, 1993; Garofalo & Lester, 1987). Similarly, more successful problem solvers classify problems based on underlying structure, while less successful problem solvers classify using surface structure (Morales, Shute & Pellegrino, 1985). Successful problem solvers plan their approach to problems carefully, and have a variety of strategies in their repertoires. According to Rogers' (1986) review, gifted students are more likely to exhibit these metacognitive behaviors than average students. Although the problem solving behaviors mentioned link closely to mathematics, she does not state what gifted construct (mathematical, verbal, or general) was used in identification, leaving open again the question about the relationship between mathematical and verbal abilities.

Problem complexity is the third theme in this literature. Students are, not surprisingly, better able to solve problems which are routine and require only a single step (Nesher, 1976; Ekenstam & Greger, 1983). Unfortunately, these are not the sorts of problems they will encounter as adults. Successful problem solvers, often talented students, express preferences for non-routine and/or multi-step problems (Garofalo, 1992, 1993), and tend to approach them in expert ways.

Understandably, perhaps, we know more about young students solving simple problems than about older students and more complex problems. In order to extend our knowledge to more complex problems and the abilities of talented students, we must examine more particular skills. The present study is intended to examine the effect of mathematical reasoning ability on problem solving when all the students are highly verbally able.

METHODOLOGY

The subjects of the study were fifty-three students (grades 3-7) enrolled in an academic year program for highly verbally able students. Students qualified for the program based on their scores on a test of verbal reasoning (the School and College Abilities Test [SCAT], ETS, 1980) administered three years above grade level. Mathematical reasoning skills on the same out-of-level test were used to divide the students into two groups, above average and exceptional mathematical reasoners.

Students were given a test of computational skills (the WRAT; Wilkinson, 1993) and three written assessments of problem solving. One problem solving task asked students to solve multi-step arithmetic problems. A second task asked students to evaluate the accuracy of proposed solutions to equivalent problems. The third task asked students to write word problems which could be solved by given multi-step arithmetic expressions. The problem solving tasks were administered in a random order to account for practice effects. The assessment scheme developed by Charles, Lester, and O'Daffer (1987) was used to score the students' solutions to problems. Holistic schemes were developed by the author to score the other two tasks. Additionally, six students were asked to complete additional problem solving tasks in an interview setting.

RESULTS

Table 1 provides score information on the computation test and three problem solving assessments. Repeated measures analysis of covariance was used for the quantitative analysis, with computation scores as covariate. There was no significant difference between the groups in problem solving, nor was there a group by task interaction. Table 2 shows the complete results.

Table 1
Mean Scores (and Standard Deviations) by Group

Group	n	WRAT	Solve Problems	Evaluate Solutions	Write Problems
Above Average	16	18.8(3.98)	16.9(9.62)	9.44(5.38)	12.8(10.3)
Exceptional	37	22.1(5.12)	22.6(7.53)	14.4(5.45)	17.4(9.41)
Total	53	21.1(5.00)	20.9(8.54)	12.9(5.86)	16.0(9.81)

Table 2
Analysis of Covariance Results

Source of Variance	df	SS	MS	F	sig of F
Group	1	.06	.06	1.71	.197
Within Groups Error	50	1.85	.04		
Task	2	.14	.07	4.50	.013
Group X Task	2	.01	.00	.16	.852
Subject X Task Within Groups Error	102	1.61	.02		

It is interesting that the more able students had a higher mean computation score yet were more likely to make computational errors. These students attempted more problems on the problem solving tasks and the WRAT. Because they attempt more problems, more able students can make more computational errors and still end up with higher scores.

Response analysis reveals several differences between the groups in approach to these items. Overall, more able students are more likely to use diagrams or charts in solving problems, to identify an error without providing an alternative solution, and to write word problems without evaluating the given expression. Less able students make fewer computational errors and more conceptual errors in solving problems. They are more likely to provide alternative solutions to problems, even when they do not identify an error. They are also more likely to evaluate an expression as part of writing a problem which is reflected by that expression. These differences between the two groups are similar to those described by Garofalo and his colleagues (Garofalo, Kroll, & Lester, 1987; Garofalo, 1993) and Krutetskii (1964/1976) between gifted students and non-gifted students. These researchers were working with students described as "mathematically talented," with little data provided to demonstrate mathematical talent or to assess verbal ability.

The exceptional mathematical reasoners were more likely to use diagrams in solving problems such as the handshake problem. The diagrams give a clear indication of student understanding of the problem situation, or lack of understanding, and may indicate that the student took the time to consider the situation as well as time to compute an answer. Organized lists of numbers were also used by exceptional students when solving the coin problem (How many ways can you make change for fifty cents?) during interviews. When writing problems, more able students were less likely to compute the value of the expression before writing a problem. Less able students were more likely to evaluate the expression without writing a problem. Even when they did write problems, the less able students were more likely to evaluate the expression, or parts of it, along with writing a problem.

DISCUSSION

Traditionally, the group differences just described are assumed to lead to greater problem solving ability. That is, students who use more diagrams, who think about understanding the problem before computing, and who understand the mathematical structure of situations are thought to be better problem solvers. The data obtained in this study are surprising because they do not support this proposition. Although the students in the more able group are older, are better mathematical reasoners, and show the differences in responses described above, they are not any better at correctly solving the given problems once differences in computational ability are addressed. A number of possible reasons for these conflicting results will be suggested in the remainder of this paper.

One possible explanation for the lack of quantitative differences is that the two groups are not significantly different in mathematical reasoning skill. The higher group had a mean out-of-level SCAT-Q score at the 67th percentile, while the lower group had a mean out-of-level SCAT-Q score at the 33rd percentile. This difference is significant at $p<0.001$ when a t-test is performed, indicating that the two groups are significantly different in quantitative reasoning ability. Even when the analysis is conducted using WRAT scores as covariate, the difference between the groups on SCAT-Q is still significant at $p<0.001$, suggesting that SCAT-Q scores are independent of computational skills.

A second possible explanation for these results is that, while the two groups have different strengths, the effects of those strengths are strong enough to keep any difference in performance from appearing. It might be that the more able group has a strength in understanding the problem, while the less able group has a strength in attention to details, including accuracy of computation. If this is the case, one might expect an interaction between group and task on the MANCOVA, with scores from the high group increasing from solving problems and recognizing solutions to writing problems and scores from the low group decreasing along the same path. There was no significant interaction between these two effects, whether or not computation was taken into account.

Another possibility for the lack of quantitative differences between the groups is that there is a threshold level of verbal ability beyond which students are equally successful at solving these types of problems. Research which suggests verbal ability influences mathematical problem solving includes students with a wide range of verbal abilities. In fact, the finding of these studies that verbal ability is a factor in mathematical problem solving is one of the reasons this study is limited to students of similar, high, verbal ability. This restriction of verbal skills, however, may have happened at a level which is high enough that all students at this verbal level are equally good problem solvers, except for their computational abilities. This would explain why the scores of the exceptional mathematical reasoners are higher, but that this variance is explained by computation. Studies using different levels of verbal ability, perhaps among students of comparable mathematical reasoning and computational abilities, would examine this possibility further.

Yet another possible explanation for these results is that the scoring system used did not allow the differences between the groups to be revealed. Computation is a factor in scoring, particularly in set A, solving problems, where it represents one third of the available points. This weight given to computation may have overshadowed differences on the understanding and planning portions of the scores on set A. Further analyses of the data did not support this possibility.

This study deliberately chose a narrow range of problem solving tasks. This was done in order to limit the sources of variance in the results, but it may have had the unintended effect of hiding the differences the study intended to illuminate. While the tasks were sufficiently difficult for the students in that scores did not approach 100% of the available points, they may not have been appropriate for revealing differences between the groups. Another aspect of this possibility is a difference in performance on routine and non-routine problems. The tasks in this exploratory study contained a mixture of the two item types, although there are more routine problems.

Further study is warranted looking at the scoring system used to analyze the data, looking at other item types, and looking at other levels of verbal ability. Additional data from these studies will add to our pool of known information and continue to define the relationship among mathematical ability, verbal ability, and problem solving.

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ONE STUDENT'S EFFORT IN RESOLVING HIS SELF-GENERATED MEASUREMENT PROBLEM: "SEE IF IT WOULD WORK WITH A TRIANGLE."

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ABSTRACT

This paper presents an example of one fifth grade student, Neal, engaged in learning as he attempted to test his self-generated conjecture in a measurement setting. Neal proposed that, as a triangle was "half a rectangle" it should be possible to use an array to determine its area, something he knew was possible with a rectangle. His intention to make sense of his various constructions in a coordinated manner caused perturbations which led to significant reorganization of his thinking. While he was unable during this forty minute session to establish the viability of his conjecture, his activity enabled him to question and further elaborate the ideas that had led him to propose the problem.

SETTING

The purpose of this paper is to present an interpretation of one person's learning as he attempted to test a conjecture he had proposed. The micro-analysis presented in this paper is part of a long term research program studying children's mathematical constructions (Reynolds and Wheatley, 1992; 1994a & b; in press; Wheatley and Reynolds, 1991a & b; 1993). Four fourth/fifth grade students participated in this phase of the study which lasted one year. Bi-weekly individual problem solving sessions, together with field notes taken in the participants' classrooms constituted the data.

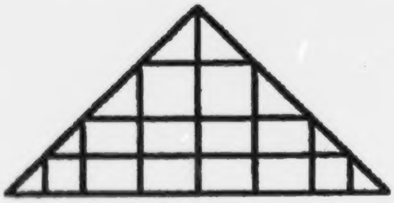
Tasks which had the potential to be problematic to the participants were presented in these sessions. Participants were encouraged to describe the ways in which they were constructing meaning for the question posed. The interviewer's stance was non-judgmental, with the participants being encouraged to act autonomously. All sessions were video recorded for further analysis by the researchers. In this paper we will analyze one session with Neal which occurred early in his fifth grade school year. On that particular day Neal was given a yellow sheet of cardboard that measured 24 cm by 32 cm and a square of red cardboard 4 cm on a side. The open-ended question posed was: What could you do with these?

ANALYSIS OF NEAL'S ACTIVITY

Neal proposed (among other things) determining the number of red squares that would be needed to cover the sheet of cardboard. This he calculated by using an array, deciding that six squares would fit vertically and eight horizontally then multiplying 6 by 8 to get an answer of 48 red squares. Neal confidently and competently explained his reasons for the viability of his solution. It is what happened next in the session that is the focus of this paper. Neal paused for a short time and then reached for pencil and paper as he said: "See if it would work with a triangle." His need to justify his previous procedure had encouraged him to reflect on his constructions (Wheatley, 1991; 1992) and propose a "wonderful idea" (Duckworth, 1987). The original question had purposely been worded vaguely so as to encourage Neal to propose a task that was meaningful to him. He chose to respond to the task in this manner and in so doing proposed a task that was challenging to him. Over the next forty minutes, as Neal attempted to elaborate his idea he created several perturbations for himself which resulted in significant learning. Neal's stance on this occasion was in contrast to his activity observed at other times, particularly in the classroom setting, when he simply chose a procedure to answer a particular, more structured question, remaining matter-of-fact and unenthusiastic in thinking about the task. Whereas on other occasions he had responded to the task with a procedural orientation, on this occasion he 'came alive.'

For Neal a triangle was "half a rectangle." By this he meant that a rectangle could be partitioned into two triangles along its diagonal. His image of a triangle being constructed in this way had been referred to on previous occasions and earlier in this session. His initial thought ("See if it would work with a triangle") was to see if a triangle could be subdivided into smaller squares (as he had done with the rectangle). In the process of experimenting with this idea he formed a conjecture about the area of a triangle, which he tested in various ways. He developed the notion that, as a triangle was "half a rectangle," then it should be possible to calculate the number of units ("small squares") in a rectangle by using his multiplicative strategy and then halve that number to determine the number of squares in a triangle formed by partitioning the rectangle along its diagonal. In Neal's activity we have identified two key moments, each of which indicates an elaboration of his argument. We will discuss his activity, using those two key moments as our focus.

First moment: His first action was to draw a triangle and subdivide it as shown below:



Neal's initial drawing of a triangle subdivided.

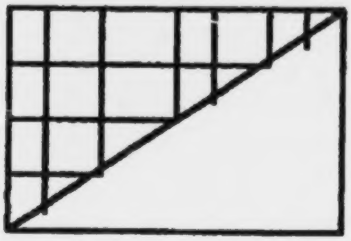
He counted eight regions along the base and four along the center vertical line segment. Multiplying these two numbers gave him thirty-two; thus he should be able to count thirty-two squares in the triangle. In this case, his idea of halving the total number obtained has been achieved through his imaging of the center vertical line segment as being half of the line segment he would have obtained if he had drawn the complete rectangle (the base of his triangle being the diagonal partitioning that rectangle to form this triangle). Neal referred to these regions as "squares." Whereas in other settings he would have clearly distinguished between the triangular and square regions, thus constructing different units, here he did not construct an iterable "square" unit by which he could somehow measure the area of the triangle. It appeared at this stage that Neal had abstracted a procedure (conventionally referred to as length by width) from his notion of an array, along with concurrent class activity of developing a procedure for measuring the area of a rectangle and attempted to apply that procedure in measuring the area of a triangle. In so doing he lost sight of the connection he had made between a concept of area as covering using iterable square units and a procedure for obtaining the number of such units that he had used in determining and defending his answer to the earlier task of finding the number of red squares needed to cover the yellow rectangle (6 x 8). No longer was he using an iterable unit in measuring the area of the triangle; he was simply attempting to apply a procedure which has become disconnected from the roots from which it grew.

His count of the "squares" in the triangle was twenty, which seemed to disprove his conjecture. Once again he was counting regions and not units; for this purpose the triangle was the same as the square.

Notice that his grid was oriented horizontally and vertically according to where he sat in relation to the triangle he had drawn. Constructing a viable relationship between his grid, his triangle and the square of which it was half became one of Neal's significant struggles during the session. He subsequently made several attempts at orienting his triangle in order to segment it into small squares in a viable manner.

After counting twenty "squares" in his triangle Neal said: "Nope", meaning that he could not accept his conjecture as viable. However, his image of a triangle as being "half a rectangle" was strong and he was perturbed by failing to get the same number in each of two ways. He examined his drawing again, rotating it 90 degrees counterclockwise as he did so and said: "It could work with a triangle maybe." He extended his triangle into the rectangle he was imaging and extended the grid lines down and across so that they now encompassed this rectangle. Notice that he had not changed the orientation of his grid, however. He then counted eight "squares" along each diagonal (the original line segments he had used in determining 8 x 4) of this square and computed 8 x 8, a total of 64 squares: "Half that is thirty-two, what we got last time." However, he persisted and counted the total number of regions ("squares") in his large square, obtaining forty-one regions. How he obtained this forty-one was difficult to determine; however, this number caused another perturbation for halving that number yielded twenty and a half: "So actually there are twenty and a half squares." It is this half that became the focus of his puzzle subsequently. However, for now he decides that he has once again disproved his conjecture.

Second moment: As previously, he was reluctant to accept his disproof. He paused and examined his various drawings, focusing in particular on the various grids he had sketched in defending his procedure for determining the number of red squares in the yellow sheet of cardboard earlier, then said: "Maybe it will ? a different shape," and "It has to be like this:" drawing another grid oriented horizontally and vertically in relation to himself. "It can't be a square and have this kind of lines across it (referring to his previous drawing which is segmented along the diagonals)." It was then that he constructed a viable relationship between his grid and his triangle as he drew the following:



Neal's later drawing with grid oriented differently.

Initially he counted the regions down the left of his triangle and those along the diagonal. However, he hesitated, reconsidering his strategy. Obviously the construction of a viable relationship between his idea of an array and his notion of a rectangle being subdivided along its diagonal to form a triangle was quite problematic for Neal. He began over, this time counting the squares down (4) and across (7), but once again he was unable to accept his hypothesis because half of twenty-eight (4 x 7) did not agree with his count of individual regions (the actual number he counted was difficult to determine; he counted subvocally and was so focused on attempting to reorganize his thinking that it was not appropriate to interrupt and ask what number he had obtained).

However, this time Neal became conscious of the "half" squares along the diagonal, connecting this with his previous puzzle of "twenty and a half squares." He now began to think in terms of iterable units rather than non-congruent regions. Over the next several minutes he attempted to figure out how best to compensate for these half squares but by this time (about forty minutes after first proposing his "wonderful idea") he had so much that he had attempted to figure out that he was unable to coordinate his various constructions. He decided that it was time to stop. He brought the session to a close in puzzlement. He had not been able to prove his conjecture but he was not willing to let go of his "wonderful idea."

DISCUSSION

Because of the particular actions that Neal decided on in testing his conjecture he was unable to resolve his major perturbations in this session. However, it was a rich learning experience for him. Whereas he had been at equilibrium as he confidently determined the number of red squares he would need to cover the sheet of cardboard (the task he had set initially), he experienced several significant perturbations as he attempted to use this scheme on a triangular shape. He maintained a conviction that a triangle could be made from halving a rectangle, yet he was unable to satisfactorily orient his array to satisfy himself that he could use this multiplicative structure in determining the number of small squares in a triangle. We suggest that this disequilibrium created the potential for significant learning for Neal as he attempted to resolve his self-generated problems in this setting.

Based on our long term involvement with Neal we believe that Neal's uncertainty, or puzzlement, will be the source of significant further learning as he attempts to resolve the dilemma he has created for himself; his dilemma being that he is convinced that two triangles form a rectangle and that an array can be used to determine the area of rectangle, however, he is as yet unable to coordinate these structures in determining the area of a triangle.

We propose Neal's actions in this setting as an example of a person engaged in learning. He generated a problem which led to perturbation as he struggled with coordinating the various constructions he had made previously. He committed himself to a process of attempting to reoganize his thinking to accommodate the greater complexity he continued to construct as he further reflected on his activity and attempted to establish viable relationships among his various ideas.

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PROBLEM SOLVING USING ARITHMETIC AND ALGEBRAIC THINKING

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A set of word problems was administered to a group of undergraduate precalculus students to study their use of algebraic and arithmetic thinking as a way to reason effectively about the given problems. An analysis of the students' responses indicates that, while many of these students were able to identify the mathematical relationships involved in the problem situations, they were unable to think about these relationships in an algebraic way. These findings are related to a model of algebraic thinking developed by Sfard and Linchevski (1994).

Contrasting Word Problems

Many students progress through four years of high school mathematics without developing appropriate levels of algebraic thinking. An understanding of the nature of this thinking is an important part of algebra reform (Kieran, 1992).

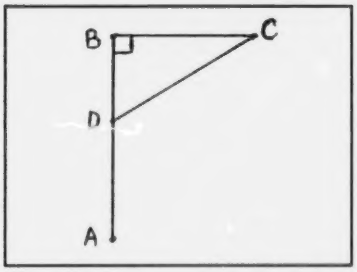
Two parallel sets of problems were developed that contained related forms of different word problems. One form of each problem could be solved using only arithmetic. The other form required algebraic thinking about the mathematical operations and the relationships between these operations. Such thinking could not be carried out unless some form of representation was used for the unknown quantities and the relationships given in the problem. Only after this representation had been accomplished could a student proceed to a numerical answer. In contrast, the arithmetic form of the problem could be solved by a series of sequential computations where operations were applied to known numbers using familiar formulas.

The questions used for each problem set are given below. In set I the algebraic problem involved understanding an income tax rate and the calculational problem dealt with a geometric figure. In set II the problem types were reversed.

Set I

1) In 1990 a family of four paid no taxes on its first $13,650 of income. All income above $13,650 was taxed at a 15% rate. If a family paid 7% of its income in taxes, what was its income?

2) (See Figure 1) Line segment AB is of length 2. Line segment BC is perpendicular to AB and has length 1. If AD is 1.3 units long, what is the sum of the lengths of AD and DC?



Set II

1) In 1990 a family of four paid no taxes in its first $13,650 of income. All income above $13,650 was taxed at a 15% rate. If the family's income was $18,500, what percent of its income was paid in taxes?

2) (See Figure 1) Line segment AB is of length 2. Line segment BC is perpendicular to AB and has length 1. If the sum of the lengths of AD and DC is 2.5, how long is AD?

The parallel problem sets were designed to act as controls for each other. Thus it was possible to test for students' understanding of the problem situation separately from their ability to use this information in an algebraic manner. When administered, the two problem sets were divided equally among 99 students enrolled in an undergraduate precalculus course. These students had previously taken three or four years of high school mathematics. The problems were given to the students two weeks before the end of the one-semester precalculus course.

Characterizations of Solution Strategies

The students' responses were analyzed by comparing and contrasting the solution strategies used for each problem. Indicators of solution strategy consisted of (i) the written expressions or formulas, in either numeric or algebraic form, that represented any of the mathematical relationships for the given problem and (ii) the uses made of these relationships to solve for the specified quantity. Student papers were separated into three categories; (1) correct solution or minor arithmetic errors, (2) evidence of some correct thinking, but containing one or two incorrect mathematical relationships, (3) no evidence of mathematical relationships appropriate to reaching a solution to the problem. Table 1 presents the distribution of student responses by category and problem type for both sets of problems.

Table 1 Solution Strategies as a Function of Problem Type

	geometric figure				income tax			
	1	2	3	total	1	2	3	total
arithmetic	47 94%	0	3 6%	50 100%	41 83.7%	7 14.3%	1 2%	49 100%
algebra	10 20.4%	20 40.8%	19 38.8%	49 100%	20 40%	16 32%	14 28%	50 100%

The distribution of responses in each category for the two problem sets shows a sharp contrast between the students' abilities to deal with mathematical relationships arithmetically and algebraically. For both the geometric figure and the income tax problem, the students (94 % and 83.7% respectively) demonstrated that they were able to understand the mathematical relationships in the given situation and correctly compute an answer as long as they could represent and deal with these relationships at an arithmetic (numerical) level. In contrast, when it was necessary to work with these same relationships at an algebraic (symbolic) level, only 20.4% and 40% of the students, respectively, were able to arrive at a correct solution. When students were denied a numerical access to the mathematical relationships, it appeared that they lacked the alternative algebraic way to think about quantity, operations, and order. It is postulated that this inability to think algebraically prevented students from understanding problems that were otherwise clear to them within an arithmetic framework.

In the geometric problem 47 of the 50 students answering the arithmetic question were able to select the appropriate relationships and evaluate these correctly. Even though these students were currently studying trigonometric relationships in the course, most of them (36 of the 47) used a more straightforward approach involving the Pythagorean Theorem with triangle BDC. When the same problem required an algebraic solution, however, only 10 of 49 students (20.4%) applied the Pythagorean Theorem to this triangle to create and solve an appropriate equation. None of the students who used trig managed to solve the problem.

Most of the unsuccessful students attempted to reach a solution using trigonometric relationships on triangle ABC. Because they could not perform direct numerical calculations on the two unknown sides of triangle BDC, they drew in and worked with triangle ABC for which trigonometric relationships do yield numerical, but irrelevant, calculations. These responses illustrate the inordinate lengths to which the students were willing to go in order to be able to find numerical computations they could perform.

The income tax question is interesting because it demonstrates students' abilities to handle a multi-step problem in which they must understand and relate two different mathematical relationships. In the arithmetic form 83.7% of the students were able to arrive at the correct answer, compared to 40% in the algebraic form. At the other extreme, only one student could not demonstrate at least a minimal understanding of the mathematical relationships involved in the arithmetic form of the problem, while 14 students were unable to make any headway when the problem was presented in an algebraic form.

The responses to the two forms of the income tax problem demonstrate the degree to which the students were competent problem solvers as long as they were able to operate arithmetically. At this level of thinking 83.7% were able to understand the problem situation, identify the mathematical relationships involved, select appropriate operations, and carry out the calculations in the correct order. However, when these same mathematical operations and order had to be represented and manipulated symbolically rather than used computationally, 60% of the students answering the algebraic from were unable to successfully solve the problem. It is postulated that these students lacked some form of mental representation for the missing numerical values that would have allowed them to reason about the problem situation. Similar behavior has been reported by Simon and Stimpson (1988) where students gained access to reasoning about unknown quantities through diagrammatic representations.

Characterization of Algebraic Thinking

A second analysis was made of the students' responses for the algebraic form of each problem in order to draw inferences about the nature of the students' algebraic thinking. This analysis focused on noting the appearance of algebraic symbols in formulas and equations and examining the uses made of these symbols. Student papers were separated into three categories; (A) algebraic symbols were used to represent equations expressing the mathematical relationships unique to the given problem, (B) algebraic symbols were used to represent familiar formulas, and (C) no literals were used - all symbols in the response were numerical. Table 2 presents the distribution of algebraic categories across the three solution strategies developed in the first analysis.

The nature of the mathematical relationships inherent in each problem situation influenced the types of possible responses. In the geometric problem many of the incorrect responses used irrelevant trigonometric formulas. Such responses were classified in category B. Since the relationships in the income tax problem cannot be expressed by any classic formula, responses to this problem tended to be classified in categories A or C. The responses in category 3-B were those for which students used an irrelevant formula.

Table 2 Form of Algebraic Response for each Problem Solving Category

	Geometric Figure (algebraic form)				Income Tax (algebraic form)			
Category	A	B	C	Total	A	B	C	Total
1	10 100%	0 0%	0 0%	10	18 90%	0 0%	2 10%	20
2	7 35%	12 60%	1 5%	20	14 87.5%	0 0%	2 12.5%	16
3	0 0%	17 89.5%	2 10.5%	19	3 21.4%	5 35.7%	6 42.9%	14

In category A students represented the mathematical relationships of the problem situation in the form of an equation in which two separate algebraic expressions containing a single variable were set equal to each other. The solution to the problem was obtained through algebraic manipulations of this equation rather than through the series of numerical computations characteristic of categories B and C. Category A represents an algebraic approach to the problem while categories B and C are essentially arithmetic.

Algebraic versus Arithmetic Thinking

Sfard and Linchevski (1994) propose a model of the development of algebra which, among other aspects, addresses the distinction between an arithmetic and an algebraic use of symbols. Based on an epistemological analysis supported by the historical development of the subject, this model presents the development of algebra as a sequence of two hierarchal levels - generalized arithmetic and abstract algebra. Within each level the perception of algebraic entities passes through an operational and then a structural stage related to the process-object duality inherent in many mathematical objects.

The categories described in the present study are found in this model within the operational and structural stages of the first level (generalized arithmetic perceptions of algebra). (Category C represents a purely arithmetic level which precedes algebra.) Category B corresponds to what Sfard and Linchevski call a numeric focus within the generalized arithmetic level's operational stage.

According to the model, students at this phase of development perceive algebraic formulas as vehicles that convey a sequence of numerical processes of which the end result is a number. The focus is on executing rather than on expressing, manipulating, and studying these processes. Student responses in category A fit a focus within the generalized arithmetic level's structural stage in which the perceptions of the structure of algebra involve variables denoting unknown but fixed values.

The nature of the problems did not make it possible to determine if students in category A also possessed a more abstract functional view of algebra represented in the model by a more advanced focus within the structural stage of generalized arithmetic. From this advanced perception, variables are viewed as representing changing (arbitrary) rather than constant magnitudes, and algebraic expressions (as opposed to numbers) are perceived as representations of the final result.

The findings of the present study support the conclusions reached by Sfard and Linchevski that the algebraic understanding of students is not very advanced by the end of their high school studies. In the present study 47 of the 99 students were classified in categories B and C, which placed them at or below the model's lowest level of algebraic thinking. Out of the remaining 52 students, only 28 were placed in category A-1 which implied an ability to reason successfully using a structural approach to algebra.

These findings do not bode well for the students' future mathematics achievement. It may be difficult to advance their level of algebraic thinking based on their previous years of traditional school mathematics. Sfard and Linchevski (p. 223) conclude that "since the functional approach is not easily accessible even for better students, it is not so difficult to understand why the mechanistic, pseudostructural approach may eventually dominate student's thinking like an overgrown weed, leaving no room for other, more meaningful perspectives."

It may be that a solution to the poor algebraic preparation of entering college students will require radical changes in the way that algebra is structured in the K-12 school curriculum. Kaput (1994) suggests a strand approach to curriculum design in which "big mathematical ideas"[1] are systematically developed throughout the complete school experience. Kaput also proposes that algebra be introduced in parallel with arithmetic as a natural outgrowth of quantitative reasoning. In this type of reasoning relationships among quantities rather than numbers and numeric relationships are of interest (Thompson, 1994). The model of Sfard and Linchevski with its stages of alternating operational and structural thought may provide the framework on which to build such an algebraic strand.

Even though the problems used in this study are similar to those encountered in traditional school algebra, their use in reformed curricula should not be rejected. Simon (1994, p. 89) points out that "activities that were used previously to "practice" a new idea may be used as situations for exploration which can contribute to the development of the idea." Problems similar to those in the study were used in the precalculus class to focus the students' attention on the differences between arithmetic and algebraic thinking.

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[1] Steen (1990) suggests the following themes as strands around which mathematics curriculum can be organized: Quantity, Shape, Uncertainty, Dimension, and Change.

Solving arithmetic word problems: Cross-national comparisons

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The objective of this study was to compare first, second, and third graders' performance in solving a set of 15 different arithmetic word problem types. A total of 82 children participated in the United States; there were 99 children in Japan, 141 children in Korea, 67 children in Sweden, and 64 children in France. All students in each country were given the same 15 problems. The problems were translated into the children's native language keeping the syntax as close as possible to the English-language version for comparison purposes.

The U.S. and Japanese data were collected first. Preliminary analyses of these data showed that there were significant differences in performance between countries and by grade within country. Japanese children performed better than did U.S. children at all grade levels, $F(1, 176) = 69.01$, $p<.001$; within countries, third graders performed better than did first and second graders, $F(2, 176) = 62.96$, $p<.001$. An examination of the proportion of correct solutions supported the expected improvement in performance by grade level.

Errors were analyzed and categorized. An analysis of errors also suggests differences between the groups. There is one major caveat that must be considered. The total number of errors varied dramatically between the two countries. By third grade, Japanese students made only 38 errors out of a possible 476 problems (8%). U.S. third graders made 133 errors out of a possible 406 problems (33%). Therefore, for the Japanese sample, the error analysis is based on only a small number of incorrect solutions.

Data from France, Korea, and Sweden are analyzed and compared in a similar fashion. The results provide an insight into the problem-solving ability of these children.

Rational Number Concepts

RESEARCH PAPERS

The Development Of Children's Concept Of Unit--Grades 4-8 35
Susan J. Lamon

Interaction And Fraction Knowledge: Children's Construction Of The Iterative Partitioning Scheme 42
Ron Tzur

ORAL REPORTS

Partitioning And Equivalence In A Geometric Setting 49
Vicki Schell and Marguerite Mason

POSTERS

Developing Conceptual Grouping Structures: New Software Approaches 50
James Kaput

THE DEVELOPMENT OF CHILDREN'S CONCEPT OF UNIT--GRADES 4-8

Susan J. Lamon
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The purpose of this study was to make inferences concerning developmental trends and needs in children's unitizing processes. Using a cross-sectional design, the study analyzed the partitioning strategies of 346 children from grades four through eight in terms of a framework that translated economy in the number/size of pieces and the use of perceptual cues into sophistication in unitizing. As grade level increased, strategies shifted away from the distribution of singleton units toward the use of more composite units and were heavily influenced by social practice related to the commodity being shared.

Research in the nonsymbolic representation of fractions (Pothier & Sawada, 1983, 1984; Hunting, 1983; Kieren, Nelson, & Smith, 1985) highlights the need for children to build a deep understanding of fractions by using a variety of concrete and pictorial models. In particular, partitioning activities form a basis for most rational number understanding (Piaget, Inhelder, & Szeminska, 1960; Kieren, 1976, 1980).

Other research investigates the role of the mathematics of quantity (Schwartz, 1976, 1988; Kaput, 1985) in the development of rational number understanding. This perspective emphasizes the importance of both numbers and their referents, that is, the linkage of units of measure and magnitude of quantities, for understanding relations and operations. This study looks at children's formation of conceptual units in the quotient construct of rational number, in the specific case of partitive division. The purpose of this study was to make children's tacit unitizing processes explicit through partitioning activities, to more clearly define the relationship between the two processes, and to connect children's partitioning and unitizing strategies to given contexts. By taking snapshots of children's partitioning strategies at each grade level, fourth through eighth, an important goal was to make inferences about trends or stages that characterize the development of increasingly sophisticated unitizing ability.

Unitizing is the cognitive assignment of a unit of measure to a given quantity; it refers to the size of chunk one constructs in terms of which to think about a given commodity. For example, given a case of cola, one could think of it as 24 cans (1-unit)s, 2 (12-pack)s, or 4 (6-pack)s. Research in several mathematical domains, including the acquisition of early counting strategies (Steffe & Cobb, 1988), the acquisition of early addition and subtraction strategies (Carpenter & Moser, 1983), the development of the concepts of multiplication and division (Steffe, 1988), and the development of ratio and proportion ideas (Lamon, 1992, 1993, 1994), suggests that the ability to form and operate with increasingly complex unit structures is an important mechanism by which more sophisticated reasoning develops.

For the purposes of this study, partitioning is defined as the determination of equal shares, and is viewed as a multi-stage operation: marking objects, cutting them, and clearly indicating one person's share. Partitioning is an operation that generates quantity; it is an experience-based, intuitive activity that anchors the process of constructing rational numbers to a child's informal knowledge about fair sharing. Unitizing is a cognitive process for conceptualizing the amount of a given commodity or share before, during, and after the sharing process. Together, the bodies of research concerning unitizing and partitioning suggest that the two processes build different and essential perspectives toward the understanding of rational numbers. The hypothesis underlying this research is that, after proceeding through the stages identified by Pothier and Sawada (1983) for representing increasingly small fractional parts, the mathematical power afforded by the notion of equivalence should help students to compose units. Growth in sophistication of the unitizing process, signified by the use of more composite units, should be reflected in students' partitioning processes.

Framework for Analysis of Partitioning Strategies

Pilot studies were used to build a framework for classifying children's fair sharing strategies. These studies revealed that strategies may be differentiated along at least four dimensions: (1) preservation of pieces that did not require cutting in cases where each person receives more than one object in a discrete quantity; (2) economy of the marking (not using sixths when thirds suffice); (3) economy of cutting (not making more cuts than necessary); (4) the nature, packaging, and social practices related to the objects being shared. In general, a higher level of sophistication in the unitizing process is indicated by the ability to use more composite units and in the context of this study, this will be indicated by the preservation of pieces that do not require cutting and economy in making necessary marks and cuts. Student strategies are of the following major types:

Preserved pieces strategy. In the case that each person will receive more than 1(1-unit) of the total quantity being shared, the student marks and cuts only the piece that requires cutting and leaves the 1-unit unmarked and intact.

Mark all strategy. In this strategy, all of the pieces are marked, even those that will remain intact, but only the piece(s) that require cutting will be cut.

Distribution strategy. All pieces of the whole are marked and cut, and the smaller pieces are distributed.

Additional refinements are possible within each of these categories. The manner in which a student marks pieces may or may not be economical. For example, a student might choose to mark a pizza with 6 equal pieces when sharing among 3 people, or might mark a candy bar into 2-piece chunks when 4-piece chunks could be used. In addition, cutting is not always executed in agreement with the marking. Thus, even when a student has over-marked, say, into 6 parts for 3 people, the pieces may be cut two together, as if the object had been marked as thirds. These additional characteristics allow for ordering of strategies within the main categories according to economy.

Method

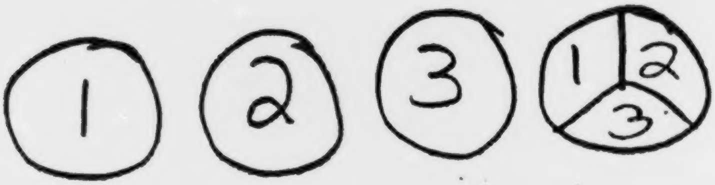
Children from fourth through eighth grade ($N = 346$; $n_4 = 63$; $n_5 = 60$; $n_6 = 72$; $n_7 = 69$; $n_8 = 82$) were given partitioning tasks designed to include several known and hypothesized influences on partitioning strategies. In addition to the usual distinction between discrete and continuous elements, tasks were differentiated according to the manner in which food items are packaged. For example, the packaging of food items may suggest or constrain certain strategies, as in the case of eggs packaged in rectangular arrays, and sectioned candy bars. Composite units such as 6-packs of cola or packs of gum may be left intact or opened to reveal individual items, etc. The children were asked to create graphical representations for each person's share of the given food. Some participated in interviews to further explain their strategies.

Results

To illustrate each of the strategies and its subtypes and to highlight some of the major results of the study, student productions for one problem type (continuous, dissectible elements) will be pictured in a hierarchical arrangement showing high to low sophistication in unitization. These strategies were applied to two problem types: (A) 3 people sharing 4 pepperoni pizzas; and (B) 3 people sharing 4 unlike pizzas, 1 cheese, 1 pepperoni, 1 mushroom, and 1 sausage. The percentages of children at each grade level using each strategy are given in the tables that follow.

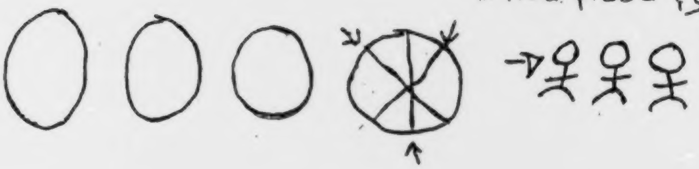
Preserved Pieces Strategy

A



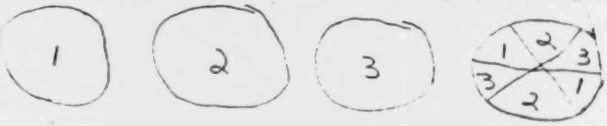
B

They would each get 1
whole pizza & $\frac{2}{6}$ ow of the othe



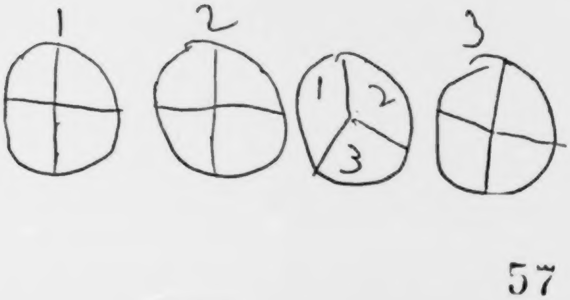
C

Each person get a whole pizza and 2 slcie



Mark All Strategy

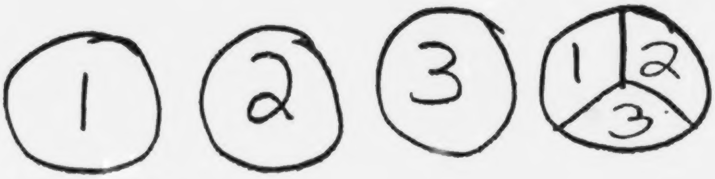
D



Each get 1 1/3 pizza

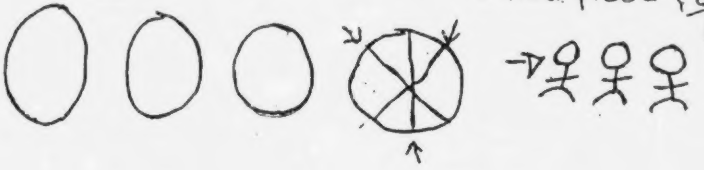
Preserved Pieces Strategy

A



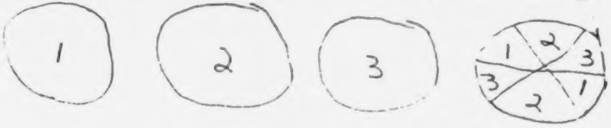
B

They would each get 1 whole pizza & $\frac{2}{6}$ out of the other



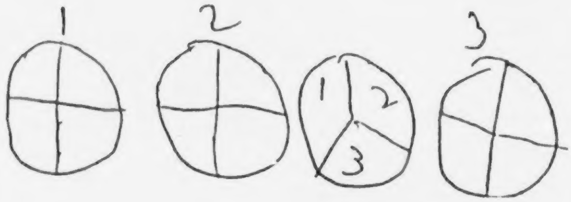
C

Each person get a whole pizza and 2 Slice



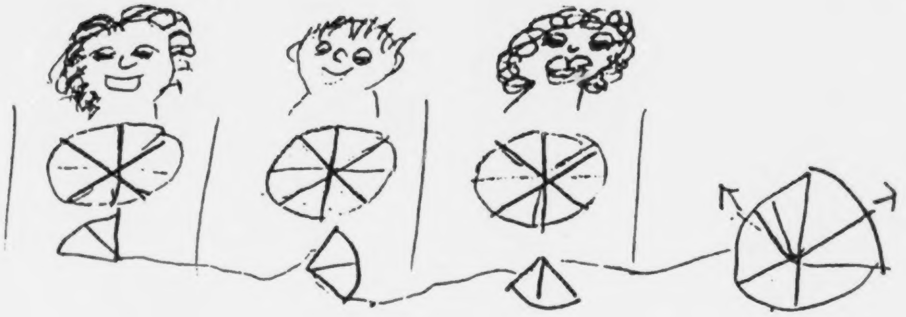
Mark All Strategy

D

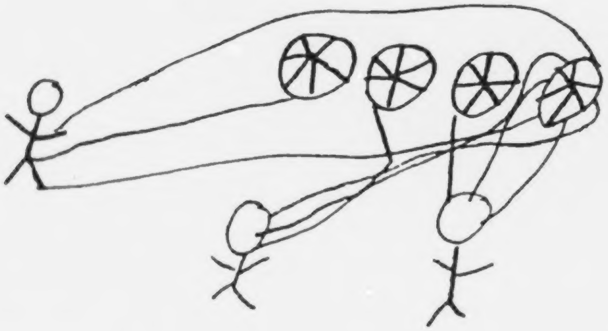


Each get 1 1/3 pizza

E

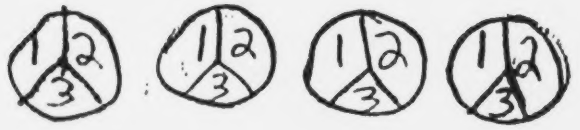


F



Distribution Strategy

G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1 2 3
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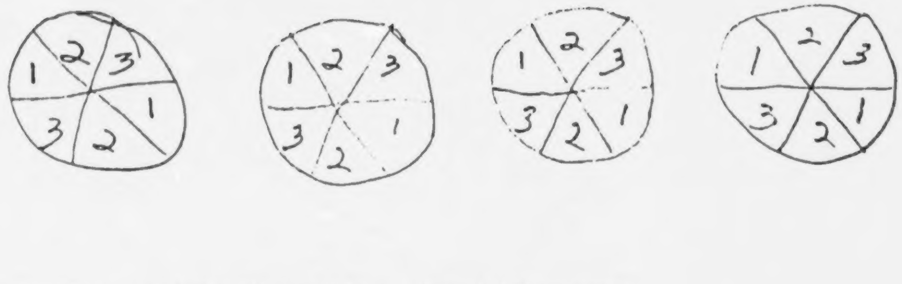
1 2 3

H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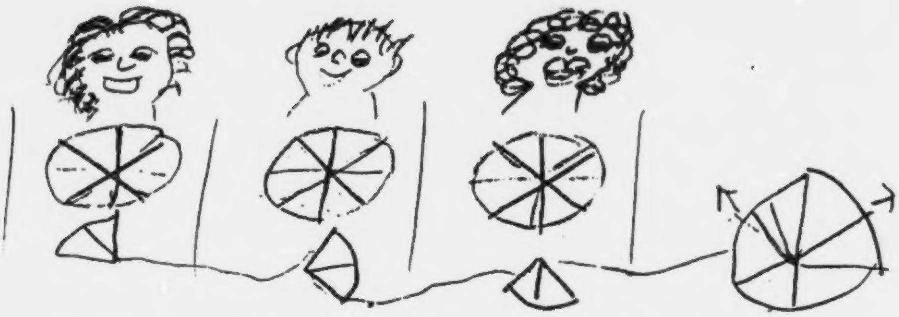
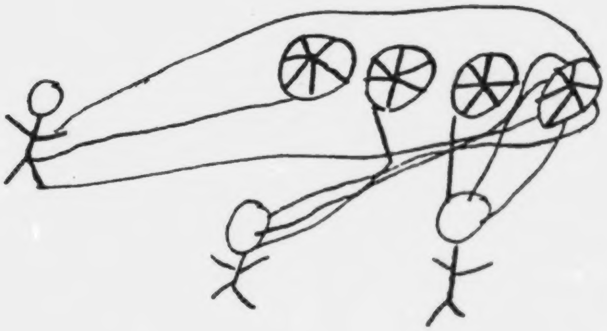
I: Incomplete, Incomprehensible, or Invalid Strategy

E

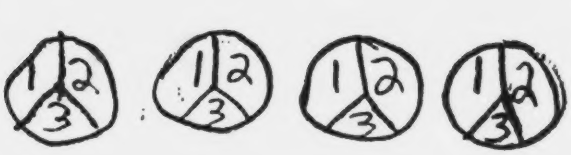


F



Distribution Strategy

G


1 2 3
1 2 3
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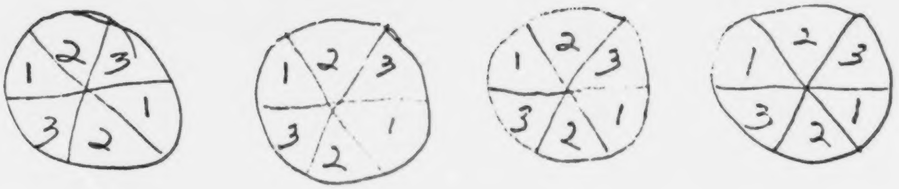
H


1 2 3 1 2 3
1 2 3 1 2 3
1 2 3 1 2 3
1 2 3 1 2 3

I: Incomplete, Incomprehensible, or Invalid Strategy

Percentage of Students at Each Grade Level Using Each Strategy for Partitioning 4 Like Pizzas

Grade	A	B	C	D	E	F	G	H	I
4	5	0	5	10	0	9	14	9	48
5	15	7	0	10	0	0	18	0	50
6	25	0	0	17	4	0	21	8	25
7	44	0	0	17	0	4	9	4	22
8	45	4	1	18	3	1	7	6	15

Percentage of Students at Each Grade Level Using Each Strategy for Partitioning 4 Unlike Pizzas

Grade	A	B	C	D	E	F	G	H	I
4	14	0	0	0	9	5	22	10	40
5	20	2	0	0	0	0	28	5	45
6	8	4	0	8	10	4	38	0	28
7	7	0	0	0	4	0	57	7	25
8	7	0	6	0	1	1	63	11	11

Discussion

Children's partitioning strategies were situationally specific and showed a strong observance of social practice and practicality. When different kinds of pizzas are shared, for example, it is customary to have some of each. Except when customary practice superseded concerns for economical sectioning and cutting, as grade level increased student representations showed movement away from decompose-and-distribute strategies toward the use of more composite units, as in the case of sharing four like pizzas. While children had no difficulty reunitizing into smaller units, decomposition was not immediately reversible. Students who had not formed cognitive composite units seemed to need the visual support of the mark all strategies. Students' written work and interviews indicated that anticipatory schemes and visual cues contributed to an operational equivalence that mediated their adoption of more composite units. The most difficult situations for children of all grade levels, were those that asked them to partition composite wholes such as packs of gum and six-packs of cola. This study suggests that partitioning activities should play a more prominent role in the middle school curriculum, and that their use should be sustained until students have developed flexibility in unitizing across a wide variety of contexts.

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INTERACTION AND FRACTION KNOWLEDGE: CHILDREN'S CONSTRUCTION OF THE ITERATIVE PARTITIONING SCHEME

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A year-long teaching experiment, guided by a constructivist framework and conducted in STICKS computer microworld, was focused on the teaching-learning interaction and the emotional component involved in two fourth-graders' construction of the Iterative Partitioning Scheme (IPS). Five steps in the elaboration of the IPS were identified in Jordan and Linda's means to make sense of fraction situations: Fractions as nominal members of a collection, fractions as countable nominal units, fractions as composite nominal units, fractions as measuring units, and fractions as composite units that can be de-composed. Many tasks and games but no manipulation of written symbols were used to enhance this fundamental fraction scheme.

Background

Personal rational number knowledge is complex and textured, and it is not surprising that the construction of fraction knowledge, developed over a long period of human history, is a serious obstacle for many children (Behr et al., 1992; Davydov and Tsvetkovich, 1991; Kieren, 1988). In other words, "the teaching and learning of fractions is not only very hard, it is, in the broader scheme of things, a dismal failure" (Davis, Hunting, & Pearn, 1992, p. 3). Back in 1960, Piaget et al. suggested seven fundamental properties related to one's conceptualization of part-whole relations, and Kieren (1976) proposed five "personalities" of fraction knowledge. Since then, many have studied the various aspects of fraction learning (see Behr at al. 1992). In the 1980s, a paradigm shift toward a constructivist view took place, which increased the number of studies on children's mathematics learning. However, these studies did not address the teaching task involved (Stephens, 1990), and were not yet embedded in a comprehensive theoretical framework. In this study, based upon an elaborated constructivist framework, the iteration method (Steffe, 1993) rather than the splitting method (Confrey, 1990) was used to initiate children's construction of fraction schemes, because this method builds upon the child's whole number knowledge (i.e. counting schemes). This method provides the child with an opportunity to construct a unit-fraction scheme in which he/she uses the common intensive meaning of 3 and 1/3—an iteration of a unit three times. In particular, three questions were addressed: In what ways did the teacher-learner interaction influence the students' construction of their IPS? What types of perturbation and stumbling blocks did students encounter? What types of modifications seem to be involved in this process?

Theoretical framework

From a constructivist perspective, knowledge is considered the dynamic product of one's active construction and the organization of his/her goal-directed activities and experiences (von Glasersfeld, 1987); Learning is considered a process from which this product partially springs, and is viewed as the capability of one's conceptual structure to change in response to perturbation. Tzur (1994) elaborated a model of learning proposed by Steffe and Tzur (in press), by identifying four routes, three loops and one assimilatory process, that link mental interaction and environmental interaction. (1) An entirely mental and personal *loop* in one's mind, in which a perturbation and a will to neutralize it is created in one's current knowledge (CK), resulting in a modification of the CK (e.g., thinking). (2) An exclusively personal but not entirely mental *loop* in which a behavior that is generated by one's CK, if assimilated back into CK, might create a perturbation and a will to neutralize it, resulting in modification of the CK (e.g., private speech). (3) An interactive, mental-environmental *loop* in which one's CK generates behavior that is directed toward people/objects in one's environment. If assimilated back into one's CK, the reaction from those people/objects might create a perturbation and a will to neutralize it, resulting in modification of one's CK (e.g., a child's click on the computer's mouse creates a mark on a stick to which the child reacts: "This is 1/4"). (4) A mental-environmental assimilatory *route* in which assimilation of other people/objects into one's CK may turn into perturbation and a will to neutralize it, resulting in modification of the CK. The major feature of this route, if it is *not* turned into an interactive loop of the third kind, is the receiver's passivity. The second loop necessitates the first one and the third loop necessitates the second and the first one. Consequently, these four types of mental-environmental interactions explain the importance of active experience to one's knowledge construction—they involve all three loops, and generate many opportunities for perturbation and knowledge modification. They also emphasize the need for delay-time, i.e., uninterrupted loops of the first type in which one's limited capability to focus attention is utilized for mental operations.

Indeed, students and teachers cannot simply construct one another's knowledge, because they have no direct access to it. Therefore, a teacher's attempt to achieve approximation between the student's CK and its plausible modification should be guided by the teacher's second order models regarding the students' zone of potential construction (ZPC) and range of bearable perturbations (RBP). The second order model refers to the teacher's view of student understanding, thinking strategies, and feelings, based upon interpretations of student language and actions. The ZPC refers to knowledge construction that is potentially realizable if the teacher were involved in student learning. Thus, ZPC is a feature of teacher-student interaction, it is not a property of the student alone. The RBP refers to the scope of cognitive activities in which a learner can tolerate the emotional discomfort resulting from the gap between one's goals and his/her current means to accomplish them. The RBP result from accumulated successful and/or unsuccessful attempts to neutralize perturbation, and determines the person's willingness to learn and the difficulty level of goals toward which his/her activities are directed.

Methodology

As a teacher-researcher, I conducted this teaching experiment in the academic year of 1992-93 with two fourth-graders for whom it was the second year of participation in a three-year research project involving 13 children, in the learning environment of STICKS computer microworld. Various operations can be performed in this microworld, given hereafter in ALL-CAPS. A stick may be DRAWN and COPIED as many times as desired. It can be MARKED vertically by positioning the cursor arrow on the stick, and any mark can be moved or erased. Any stick can be partitioned into *n/n* equal pieces by clicking PARTS, "dialing" a number between 2 and 99, then clicking on a stick. Any stick can be CUT in a desired point, JOINED to any other stick, or dragged to a desired position (including TRASH). A stick can also be REPEATED, and, if MARKED, any stick can be BROKEN to its pieces. Any number of parts can be PULLED out of a partitioned stick without changing its original appearance, and sticks can be MEASURED by COPYING a measuring-stick into the ruler. Sticks or parts of them can be FILLED with 10 different colors, LABELED by a fraction symbol, and COVERED/UNCOVERED by a rectangular shape.

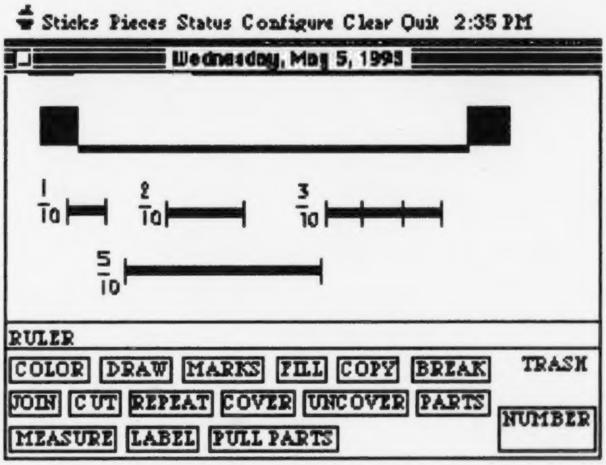

Figure 1: The computer's screen from 5-5-93.

At the beginning of the project's second year, Jordan and Linda were paired because they studied in the same class, and because they seemed to have established the Explicitly Nested Number Sequence (ENS) (Steffe and Cobb, 1988), i.e., they were using iteration of composite units. The study consisted of 29 videotaped episodes, 25-minutes each. Two faculty and 2-3 graduate students observed those episodes, and collaborated in the on-going analysis and the planning sessions for each episode. I also observed the students' mathematics class 12 times during the four weeks in which their teacher taught fractions.

Analysis/Results

Jordan liked mathematics and said that "It's fun doing math. That's my favorite subject." It was better to learn math in the microworld, since "here we don't have to do paper work." Jordan was an enthusiastic and competitive student who was particularly excited about big numbers. Linda liked the work in the computer microworld because we "learn about math and, ah ... you won't have such a hard time in class." She was usually excited in her work, especially in tasks for which she anticipated the answer, and searched for teacher's cues regarding the quality of her work.

Re-establishing the ENS. The teachers' goals in the first seven episodes were to revive the children's ENS and familiarity with STICKS microworld, and to establish a cooperative team. We used three types of tasks: (a) creating an *n*-stick, i.e., a stick consisting of *n* pieces (e.g., 100-stick, 72-stick); (b) a taking-turn game in which they built a stack of 10 unit-sticks under the constraint that one should try to make a new stick using only a single unit-stick that was previously made, preferably not the 1-stick; and (c) we all took turns in making a stick from one of the 10 unit-sticks in the stack (e.g., 24-stick), then asking the other two, who had closed their eyes, what stick was used or how many times it was REPEATED. The other two had to figure out the missing property. For instance, to find the answer for a 24-stick that I made by iterating an unknown stick three times the children tried to find "what goes three times into 24." Of course, the important part of the task was their reasoning (usually by REPEAT). I also used open-ended question such as "how many times and which unit-stick did I use to make the 30-stick?" From the children's actual and mental iteration of composite units, we inferred an underlying dual-anticipation upon which they could have constructed their IPS: The small stick must be iterated an exact number of times to exhaust the desired stick.

Establishing the Iterative Partitioning Scheme (IPS). The establishment of the IPS was initiated in sharing tasks because it is a natural activity of children, in which the notion of a "correct" answer is tightly linked with personal sense of "fair" solution. (Note: m/n expressions should be read in their name, e.g., 1/3 reads one-third.) For example, on 11-11-92 I asked them to make a 30-stick from the 15-stick. Then I BROKE it into its halves, FILLED one half in a different color and asked "How would you call the share of each one of you?" Both said: "A half." Later I asked: "You said that this is 1/2, this is 1/3, and this is 1/4. What makes it 1/2, 1/3, 1/4?" Jordan explained that it's "1/4, because it's *one-out-of-four*." The children's response "one-half" and Jordan's comment "it's one-out-of-four" provided us with a way of communication and highlighted that the specific meaning of an entity that resulted from the iterative method is determined by the context . Hence, a 10-stick that was iterated four times is one-fourth of a 40-stick because "*it's one-out-of-four*," while the same 40-stick, if created by iterating a stick four times, was done from the 10-stick because "*ten goes four times into 40*."

A set of sharing tasks was performed in a continuous situation—sharing a long and undivided stick among two, three, four people, etc. The children DREW an estimated 1/n-stick, then REPEATED it *n* times to see if it was equal in length to the original. Another set of tasks involved estimation of a fractional part of a whole. For example, on 12-7-92, I prepared seven non-labeled fraction sticks (1/2 through 1/8) and they had to estimate how much of the whole was each stick, using REPEAT (or COPY-JOIN) to verify their answers. In doing this, they showed their understanding that the 1/n-fraction unit has two interrelated properties—all parts are equal in length and together they re-compose the whole. The last set of tasks before introducing PARTS involved sharing a whole into an *n/n*-stick by estimating the location of only one mark for a desired 1/n fraction (e.g., 1/5).

Elaborating the IPS. The potential actions provided by the computer microworld cannot transform people's mathematics. Yet, when a person's conceptual structure incorporates technological "short-cuts," they become powerful tools for thinking-communicating-doing mathematics. PARTS is such a tool, and was added to our teaching-learning enterprise by free-play, after the children showed their ability to anticipate the iterative part-whole relation. Then, five stages were identified in the construction of the IPS.

The IPS's first stage: Fractions as nominal members of a collection. Here, the child's conception of fraction is as a unit that belongs to a specific and independent compound collection ("family"). The members of this collection are *d* identical parts, each of which can be iterated to re-compose the collection and is considered *"one-out-of-d" parts to which the reference whole is evenly partitioned.* Each of these *d* unit-fractions receives its collective identity (i.e., "last-name"), from the number-word that is coordinated with *d*. Thus, one-eighth is the "last-name" coordinated with any unit that belongs to a closed and independent family of exactly 8 members. For example, when I asked Linda: "Nine people came to your party, and I'm thinking of a stick that is your share," she used PARTS(9). Then I asked "how much is Linda's share out of the whole cake?" Both immediately said "one-ninth, because it's *one-out-of-9*."

The IPS's second stage: Fractions as countable nominal units. Here, the child's conception of a fraction is as a countable object (unit) within the fraction's distinct and *bounded* "nominal-family." The child applies his/her counting schemes *only within* "one-out-of-*d*" distinct collections. Thus, fractions of the type n/d ($n \leq d$) receive their collective identity (i.e., "last-name" or "denominator") from the number-word that is coordinated with *d*, and their individual identity from the number-word that is coordinated with the number of iterations *n* (i.e., "first-name," hence numerator). For example, on 12-7-92, Linda REPEATED a stick six times to prove that Jordan's estimation of it as 1/6 was correct. Then Jordan FILLED 2/6 in different color and said: "It's 2/6." I asked Linda if she could show 5/6 and she used FILL to make them, but to my question "Can you make 7/6?" both said "no." Thus, the biggest possible fraction of a "family" in the second stage is *d/d*, since exceeding this number, say 12/10, transforms the 10/10-whole into a new collection consisting of 12 rather than 10 parts.

The IPS's third stage: Fractions as composite nominal units. Here, the child's conception of a fraction is extended to include iteration of composite units, still within the *d* whole numbers of the family. Of course, just as in whole numbers, confusing addition and multiplication is expected. For example, I thought of "a stick that is twice as long as the 3/11." Jordan COPIED two more 3/11-sticks, JOINED them and said it was 6/11. I asked why and he explained "Because 3/11 plus 3/11 is 6/11." Linda said she thought of it as 5/11 because "you said twice as long, therefore it's 3/11 plus 2/11."

The IPS's fourth stage: Fractions as measuring units. Here, the child's conception of a fraction breaks through the boundaries of the *d* whole numbers ("family") to which a fraction belongs. This is a qualitative transformation, since the fraction is no longer perceived on a nominal basis, but rather as a unit of measure—a relation. Now, it earns its collective identity ("last-name") from the *explicit* conception of each part as a *measuring unit with fixed relation to a certain measured unit* (reference whole). Thus, 24/12 can be perceived as a result of 24 iterations of a simple member of the family 12 (i.e., all the units that keep the 1/12 relation to a reference whole), or as a result of three iterations of the composite unit 8/12. On 12-7-92 ("Can you make 7/6"), on 3-8-93 ("Can you make 6/5"), and on 3-10-93 ("Can you make 10/8") they had always said "No," indicating their inability to make sense of improper fractions. On 3-31-93, they overcame this stumbling block. I thought of "a stick that is twice as long as the 6/11." Linda COPIED the 6/11 twice and Jordan said that "there are 12 pieces, as if 12 came to the party—therefore it is 12/11."

The IPS's fifth stage: Fractions as composite units that can be de-composed. Here, the child can perform a reverse operation in which any composite fraction *n/d*—first proper, then improper— can be de-composed to *n* parts of *1/d*, each of which can be iterated to re-compose the whole (*d/d*) or any other number of parts. For example, at the beginning of the 27th episode (5-5-93), four sticks with corresponding labels attached were presented: 1/10, 2/10, and 5/10 (unmarked) and 3/10 (marked, cf. figure 1). Our hypothesis was that the children would begin to re-compose the original whole from the 5/10, then reflect on this action and use the 1/10 and the 2/10, and finally struggle with the problematic situation of the 3/10. In this exact order, and with great difficulty in re-composing the whole from 3/10, they solved all four problems.

Discussion

The teaching-learning interaction concentrated on elaborating the children's conceptual structures, and was based upon on-going analysis of their thinking strategies, demonstrated by their language and actions. We strove to create interaction in which various playful activities turned our mathematics learning into joyful experiences. In a period that was equivalent to 3.5 school-weeks, Jordan and Linda constructed the five stages of the IPS on the basis of their Natural Number knowledge and the iterative method, as a means to cope with and deeply understand part-whole situations. Using no arithmetic manipulation of written symbols, both children demonstrated a powerful capability to perform addition and subtraction of proper and improper (same-denominator) fractions, as well as to re-compose the whole from its part. It should be noted that the cognitive-social-emotional process and the five steps identified were not accomplished in a linear, straightforward manner.

Two major implications of this study are of importance to teachers' work in the classroom setting. First, the five stages should guide, in a conceptual rather than symbolic manner, the ways in which teachers try to be involved in their students' learning of fractions. In particular, teachers should be aware of children's difficulties in transforming the nominal conception of fractions into a notion of relation. Second, teachers can make use of the many tasks and problems used in our teaching experiment. Though highly recommended, not every classroom teacher has access to computer microworlds such as STICKS. Nevertheless, the tasks, questions, and games can easily be modified and used with simple manipulatives, as long as the teacher maintains the conceptual underpinnings of these activities.

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PARTITIONING AND EQUIVALENCE IN A GEOMETRIC SETTING

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Theoretical Rationale

It has been suggested that, in the development of rational number concepts, partitioning and equivalence serve as "constructive mechanisms". This paper describes students' partitioning of unit rectangles to obtain equivalent parts, and describes thinking strategies students employ as they develop understanding of the concept of equivalence.

Method

Three hundred students, from fourth graders through college students, were asked to partition a rectangle. Additionally, twenty students were interviewed as they worked in a cooperative setting. The first level of analysis consisted of ordering the partitions according to frequency of appearance, then classifying them into categories. Second level analysis consisted of determining the hierarchical pattern of the categories along a continuum.

Results

Categories of partitions were identified, and corresponding frequencies indicated the universality of these partitions. Clusters of strategies displayed by students in completing the partitioning were: Complete Visual ("equal parts" at this stage are limited to congruent pieces); Combination of Visual and Deductive (students at this level may begin with congruent pieces for the initial cut, but then can use compensatory techniques or justifications ; Deductive (students at this level use mathematical thinking and area formulae to justify equality).

The most common partitions among all age levels were partitions which could be justified on a strictly visual basis; the partitioning mechanisms employed were horizontal, vertical, or diagonal cuts. Students next created partitions which still required visual justification, but utilized mechanisms more complex than the primitives. Students who are functioning at the deductive stage can construct partitions in which none of the pieces are congruent, by means of a mathematical justification.

Poster Session

Developing Conceptual grouping Structures: New Software Approaches

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We will demonstrate and discuss with interested parties a new series of software environments intended to help students in grades 2-6 develop the conceptual grouping structures underlying understanding of the number system. The usual approach is to use strictly hierarchical groupings that mirror the exponential structure of the base b number system. The alternative presented here is to support a much more flexible grouping environment, linked to more formal notations, that allows the student to develop a sense of grouping groups in ways that are much less formal than normally experienced. For example, students can take a loose collection of screen objects, say candies, and group them into packages of 3 each, package these into bags of 5 packages each, package the bags into boxes of 4 bags each, and put the boxes into crates of 12 boxes each. They can trade these across packaging schemes, and create new organizing schemes. We will discuss various activity structures and, for those who are interested, make the software available for testing with students.

Research Methods

RESEARCH PAPERS

Using Case Studies To Promote Instructional Change 53
Barbara Armstrong, Judith Sowder, and Randolph Philipp

The Validity Of Concept Maps As A Research Tool In Remedial College Mathematics 60
Jan Laturno

Collaborative Action Research In Mathematics Education: A Tale Of Two Teacher-Researchers 67
Anne M. Raymond

USING CASE STUDIES TO PROMOTE INSTRUCTIONAL CHANGE

Barbara Armstrong, Judith Sowder, and Randolph Philipp

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This investigation examined teachers' perceptions of case studies as change agents for reform in mathematics education, and teachers' views on ethical issues related to case study research. Seven elementary teachers each read from two to four case studies, reacted to them individually, then met as a group with the investigators to discuss them. The teachers felt that the studies justified their own struggles to improve their teaching of mathematics, but worried that they could never attain the mastery of some of the case study teachers. They thought that researchers needed to be more sensitive when criticizing teaching behaviors, but ethical issues did not seem to be of great concern to them.

Kagan (1993) has described three uses for classroom cases: "as instructional tools to help novice teachers connect theory to practice and develop problem-solving skills; as raw data for research on teacher cognition; and as catalysts that can promote change in teachers' pedagogical beliefs and practices" (p. 704). In this investigation, we focus on the third use: promoting teacher change. Our primary purpose for undertaking the investigation reported here was to examine teachers' perceptions of the value of case studies as change agents for the reform of mathematics instruction.

Using cases as a way of studying educational phenomena is not new, but we in mathematics education have only begun to use case studies as a way of coming to a deeper understanding of the teaching process. In part, this delay is due to the fact that qualitative methods of research have gained widespread acceptability only in the past decade or so. We believe that the recent increase in numbers of case studies in mathematics education is also due to the need for information on the success of efforts to reform the teaching of mathematics (see, for example, Fennema, Franke, Carpenter, and Carey, 1993; Cohen, 1990; and the several other case studies in the California study of elementary mathematics, Peterson, 1990).

A secondary purpose for our study was to obtain views of teachers on ethical issues related to case study research. In particular, we were concerned with the effects of negative reports about the teachers who consent to be part of a case study. What is the researcher's obligation in such cases? Should reports that might be perceived as negative be shared with the teachers studied? We discerned that such issues are of concern to mathematics education researchers, based on the large number of researchers who elected to discuss ethical issues at a discussion group session we attended during the 1993 PME-NA.

Ethical issues have, of course, always been of concern to mathematics education researchers. But qualitative methods have changed the ways in which we think about ethics in our research. "For many qualitative researchers, ethical questions do not reside narrowly in the realm of how to behave in the field. Rather, ethics are understood in terms of their lifelong obligations to the people who have touched their lives in the course of living the life of a qualitative researcher" (Bogdan & Biklen, 1982, p. 55). Brody (1981) argues that the process of making ethical reasoning conscious and explicit can serve to improve the ethical reasoning that we undertake. Anonymity is especially difficult in case study research. Many people, including the teachers being studied, know of the research as it is being undertaken and can connect published research reports with the individuals studied.

In this investigation, several teachers read case studies of teachers of mathematics, independently reflected and commented on the studies, then met as a group, with us, to discuss what they found to be of value in this exercise. They also shared how they felt about issues of anonymity and of sharing case studies with the teachers being studied. We were, in addition, interested in receiving feedback on the value of three case study reports we had recently completed.

The case studies

Four case studies were read by these teachers. Three were of teachers who were part of our research project undertaken to investigate the relationship between knowledge of mathematics and classroom practice. We wrote "Instructional Effects of Knowledge Of and About Mathematics" to illustrate the importance of mathematical knowledge in teaching middle school mathematics in a way consistent with current reform efforts. "Orchestrating, Promoting, and Enhancing Mathematical Discourse in the Middle Grades" uses the standard on discourse (NCTM, 1991) as a framework for describing the rich discourse found in a suburban fifth grade classroom. "A Responsible Mathematics Teacher and the Choices She Makes: A Case Study" describes a teacher facing all the problems of a restructured, inner city middle school. A fourth case study, *A Revolution in One Classroom: The Case of Mrs. Oublier* (Cohen, 1990) was provided as a contrast to the three essentially positive studies described above. Since we wanted to determine the teachers' views on the usefulness and ethics of the written case studies of teachers, we included a case study of a teacher who viewed herself as successfully implementing reform in her classroom, a view not held by the researcher, who concluded that the teacher's reform attempts were unsuccessful.

The selection of the teachers

We wanted teachers who were aware of the reform movement in school mathematics, who were willing to undertake the task of reading and commenting on case studies (the remuneration was small), and who would be open with us in their comments. One of us had had contact with the teachers at a restructured elementary school and felt the teachers exhibited the characteristics we were seeking. Six teachers and the principal agreed to participate.

The procedure

The teachers read from two to four of the case studies. The forms of the responses varied from notations written in the margins of the case studies to two-to-several page responses. Some of the teachers reacted to each of the studies they read separately, while some integrated their views. Some responses were guided closely by a set of questions provided as a guideline; others touched on the questions only peripherally and were more personal. Because of the holiday season, the teachers did not share their reactions with one another. When classes were again in session, the teachers met with the researchers to discuss their reactions.

Results

Most of the comments of the teachers were positive; they enjoyed reading the case studies and learned from them. The studies justified their own struggles to improve their teaching of mathematics: "What I found of value for me is that I am on the right track." "I believe and try and practice what is being said in the study." "I was constantly comparing myself to the teachers in the articles." They found most value in the actual descriptions of classroom practices, particularly the protocols of classroom discourse: "I loved the dialogue between the teacher and her students. This part was real for me, and I could apply it to my life in the classroom and see from her methods how to alter my discussions with students." "I learned how discourse promotes learning and helps make mathematics relevant."

Case studies of teachers speak to teachers where they are. Reactions to a particular article differed depending upon the extent to which the teachers felt the study's message was relevant to their own teaching. Most teachers found the case study that focused on discourse to be valuable: "I'm left with the feeling that this (classroom discourse) is one of the most important elements of teaching math." Yet one teacher said, "I hear this message loud and clear already." For her, this study "just scratched the surface, when we need to be drilling for oil."

All of the teachers expressed a general feeling of frustration associated with their perceptions of the strong mathematical background of the three strong case study teachers. All said that the case studies had made them realize that a strong mathematical background was necessary for teaching well. Their frustration came from not knowing what to do about their own weak backgrounds. One teacher put it well: "After reading about Phil, I analyzed, 'Could I teach my kindergartners to READ if I myself did not? Then why did I try to teach fourth graders long division when I really did not *understand* it myself?' The realization makes me fearful. How can I ever re-learn and re-program myself?" This frustration was strongly expressed in the group meeting. The teachers felt that inservice programs offered by the district do not meet their needs--they are too brief, and focus on such topics as how to use manipulatives rather than on helping the teachers develop a deeper mathematical understanding. Neither do universities offer the kinds of courses they need.

The teachers expressed a fear that they could never be as good as the good teachers in the case studies. They read about the length of time these teachers had been "in the process of changing," the number of projects with which they had been involved, and the support these projects had provided, and felt they could never get there, particularly since they also taught other subjects. In the group discussion, however, they were reminded that

Phil, one of the strong case study teachers, had said that on some days "when the kids are climbing the walls, or when I'm tired because I've put all my energy into a great art-lesson, I'm just not up to teaching a great math class, so I assign them something out of the book that day." Discussing their frustrations and doubts about themselves appeared to help. They decided that the fact that it took so long to become a good teacher was actually in their favor; they could not be expected to make major changes over night.

The discussion of ethical issues was primarily prompted by the Cohen case study. One said, "I felt uncomfortable reading many parts of the case study of Mrs. O. I felt that the humor was very much out of place, and sarcastic." There was general agreement in the group discussion that Cohen could have shown more respect for Mrs. O in the ways in which he chose to describe her and her teaching. However, the teachers also felt that this was the most useful of the case studies: It made a "valuable policy and practice connection." One teacher "could feel and understand the dilemma facing changing our educational system and wanted to take part in the finding of a possible solution." Another said, "I was surprised that an author would paint such a true picture of a teacher while showing the necessary changes needed in our system." One teacher shared that she would be frightened to have Cohen observe her but at the same time would love it--if no one would know her name.

The teachers agreed that anonymity was an issue, because so much specific information is provided about the teacher in a case study. This concern was particularly evident in their statements that they did not want to be subjects of case studies, at least not yet! One of the teachers knew one of the case study teachers, and said she would have preferred not to know about whom she was reading. Another teacher said that she would welcome feedback, but did not want someone observing her who did not share her teaching philosophy. Yet another stated that she did not believe there would be ethical problems if the teachers were willing participants and were given some editing rights.

Discussion

The discussion will focus on (a) what researchers can learn from these teachers about writing case studies that will be of value to them, and (b) ethical issues related to doing case studies of teachers, based on teacher comments and on our own discussions, reading, and reflections.

The teachers' comments provided insight into what they found compelling about the case reports. They were not interested in long reviews of literature that would situate the study in the research; rather they situated the study in relation to their own classroom experience and professional development. It was clear from the discussions that all of the teachers had thought about mathematics reform. Case studies in which the interaction of strong content knowledge and expertise in pedagogy are portrayed in real settings can provide a more complete picture for reform-minded teachers. They surprised us with their reactions to the case study of Mrs. Oublier because, even though they perceived a negative treatment of the teacher by the author, they were able to separate the tone from the message. They felt that the case was meaty, tightly written, and gave them something to which they could react. As a result of reading this study, one teacher reflected on her own attempts to implement reform. She asked herself if she was truly changing or merely going through activities that looked better and seemed different, but did not result in real change.

This group of teachers dispels the belief that teachers do not want to make significant changes in their classrooms and in themselves. They are asking for support in the way of inservice and graduate education that will help them achieve the levels of knowledge demonstrated by the exemplary teachers. They are also asking for realistic stories of teachers in the process of attempting reform. One teacher commented, "Personally, I would like to see more case studies, both with good teachers and bad ones, to acquire an understanding of where I fit into this crazy enjoyable world of teaching." The question for researchers is how to ethically gather the data necessary to write the stories while protecting the identity of teachers who voluntarily open themselves to scrutiny.

If mathematics educators hope to use case reports of teachers as vehicles for reform, then there must be certain premises from which we operate. A respect for the teaching task and how it differs at various levels is important. An empathy must come through for the person who engages in the task and who endeavors to do her/his best even when her/his ideas about what is best differ from those of the researcher(s). Anonymity must be protected. Teachers, schools, and school districts are very sensitive about nonconstructive negative criticism and will choose not to participate in any case study research if they perceive the possibility of such criticism. The purpose for the case study should be clearly stated along with the researchers' point of view and bias on whatever issues are raised. In the end the ethical decisions must rest with the researchers, but we are part of a community of mathematics educators that includes the teachers about whom we write. The words we use and the attitudes we portray can have the power to provide either cohesion or divisiveness.

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THE VALIDITY OF CONCEPT MAPS AS A RESEARCH TOOL IN REMEDIAL COLLEGE MATHEMATICS

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This study examined the use of concept maps as a research tool with mathematics students within a narrow range of competence. Two areas of utility were examined: correspondence between concept maps and clinical interviews in determining concept connectivity, and ability of concept maps to predict academic achievement. 118 students developed concept maps from a list of mathematical concepts. These were later correlated with their progress through remedial math courses. Interviews were conducted with 24 of the students from high and low scoring concept map groups. Placements into high and low groups by an independent researcher, based on these interviews, were compared with concept map scores. There was significant agreement between interviews and concept map scores, and significant correlations between concept maps and academic progress through the course. We argue that this is due to the degree of concept connectedness required by the activities.

The NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989) emphasize the need to connect concepts in mathematics both within the discipline and across disciplines. A student's ability to make these connections is essential in their efforts to understand the subject in a meaningful way and are demonstrated through their capacity to apply information to novel problem solving situations. However, the degree of a student's ability to connect concepts is difficult to ascertain from most exams or assessment instruments.

Concept maps are intended as a graphical representation of the psychological structure of knowledge within the subject producing the map (Novak & Gowin, 1984). They were originally developed at Cornell University by a research team desiring a concise way of representing conceptual changes that were taking place in elementary-school science students being interviewed by the team during a longitudinal study (Novak, 1990). As new researchers joined the team, they found the concept maps useful in their graduate courses as organizational methods for aiding understanding of new subject matter. Since 1975, a great deal of work has used concept maps as instructional aids within the classroom to help students learn how to learn meaningfully (Novak, Gowin & Johansen, 1983; Okebukola, 1990). Recent work with concept maps has been extended to their use in teacher education (Beyerbach & Smith, 1990) and instructional design (Starr & Krajcik, 1990).

Work with concept maps as research tools in the classroom setting has concentrated on academic comparisons between experimental groups receiving instruction in the use of concept maps and control groups (Fisher, 1990; Jegede, Alaiyemola, & Okebukola, 1990), or comparisons between student-generated and expert-generated maps (Beyerbach, 1988). We found one study comparing results of concept maps with clinical interviews after an instructional intervention in science (Wallace & Mintzes, 1990). The results of the study suggested a "strong relationship between the complexity of cognitive structure as revealed in concept maps and the extent of biologically meaningful knowledge possessed by the learner" (Wallace & Mintzes, 1990, p.1045). However, the small sample size limited the analysis.

References to use of concept maps in mathematics education research are minimal. Novak and Gowin (1984) illustrated examples of student constructed maps for mathematics in the appendix of their seminal work *Learning How to Learn*. Bright, Friel & Berenson (1993) have used concept maps with teachers to ascertain how they related various aspects of statistics.

The following study was designed to examine the ability of concept maps to differentiate the conceptual knowledge of students whose mathematical background is within a narrow range. Clinical interviews are useful for this type of research, but are extremely time consuming and generally must be restricted to a limited number of the participants in the study. Concept maps can be completed and scored in a far shorter amount of time with a wider range of participants. This study examines comparisons between results obtained from clinical interviews and concept maps. The study also examines the ability of concept maps to predict academic achievement in a mathematics course.

Methodology

Subjects

The subjects of this study were drawn from a community college. The population of the college was predominantly minority and lower S.E.S. Students from three remedial arithmetic classes and two remedial elementary algebra classes, all taught by the same instructor, were selected. There were no controls on who enrolled in the classes. All students were informed that concept maps would be required of them as part of their course work. The classes initially consisted of 202 students, but had dropped to 120 by the time of the initial data collection in the sixth week (remedial math courses at this institution traditionally have a 50% drop-out rate). Subsequently, 2 more students did not finish the classes and were dropped from the study, leaving complete data on 118 subjects.

Research Design

Students in the five classes were required to construct two concept maps during the fifth and sixth week of the semester long courses. Subsequent concept maps were also required of the students during the eleventh and sixteenth week of instruction, but are not included in the results of this report. Scores on the mathematical concept map allowed placement of students into three groups: high, medium, and low. Twenty-four students (six each from the two courses within the high and low scoring groups) were randomly selected from the groups and requested to participate in interviews during the sixteenth or seventeenth week of instruction regarding their connections of mathematical ideas. The interviews were taped and work constructed by the students during the interview was preserved as data. Data concerning how far the students progressed through the units of the course was also collected as a measure of academic achievement in the course.

Experimental Procedures

Concept maps

All five classes were giving instruction during one class period in the use and construction of concept maps during the fifth week of the semester according to suggestions by Novak and Gowin (1984). They were allowed to practice map construction in class with a written selection concerning an environmental issue. Various maps created by the students were discussed, comparing styles of graphical presentation, as students were introduced to the method that would be used to score them. Students then received a second written selection about another environmental issue for concept map construction, which was assigned as homework.

Next class period, students were encouraged to examine each others' maps as examples created from the written selection by the instructor to show variations in complexity of maps using similar concepts were available on transparencies for class critique. In order to allow students the freedom to critique the examples, the transparencies were represented to the students as having been created by students from another institution familiar with concept maps. The concept map examples were each scored by the class, with more connected, hierarchically designed maps being emphasized as receiving higher scores. The students were given the opportunity to rework their environmental concept maps before turning them in the next week. The students were then given a small list of mathematical concepts (five) to create a concept map in class. These creations were again discussed as a group and scored. Students were then given a list of ten mathematical concepts (which should have been familiar to them based on their prior course work) for construction of a concept map as homework. The students were encouraged to add concepts as they thought appropriate. Both the second environmental and mathematical concept maps were collected at the beginning of the next class meeting.

Interviews

ASPECTS OF CLINICAL INTERVIEW

	Object	Function
Part 1	Three open-ended mathmatical problems requiring use of concepts illustrated in prior concept map	Used to determine if students would spontaneously connect concepts when solving rich, ill-structured problems
Part 2	Questions regarding how the students could check their work on the problems in part 1	Used to purposely generate connections between operations and the concepts from students
Part 3	Construction of relationships between four concepts presented sequentially on index cards to the students	Used by the principal researcher to verify knowledge of mathematical concepts and the propositions that relate them

Table 1: Three partsof the clinical interview and and their corresponding function

The interview consisted of three parts, each representing separate functions, as described in Table 1. The open-ended questions were selected from the Sampler of Mathematics Assessment - Addendum developed by the California Assessment Program. They were chosen as examples of remarkably rich problems requiring thoughtful explanations by the students that would tap some of the ideas appearing on their mathematical concept maps. Students were interviewed before or after class hours in their unoccupied mathematics classroom, an environment they were very comfortable in. The students were asked to show their work on paper as they explained how they solved the three open-ended problems. They were encouraged to refer back to their work as they answered questions in part 2 of the interview.

Course Progression

The classes consisted of individually-paced units requiring mastery of material using both well- and ill-structured problems that utilized rich thought processes not traditionally tapped in the students' previous math courses. Students also were required to maintain their connections to previous units by completing 4 or 5 cumulative reviews requiring a 90% accuracy rate. The students were allowed to repeat units, using different assessment questions, as many times as necessary for mastery, with all results being discussed one-on-one with the instructor. Errors in solving problems were mutually discovered by the students and instructor by closely examining the students' work completed during the unit. The number of units mastered during the semester by the student was recorded.

Data Analysis

Concept Maps

Both of the submitted concept maps were analyzed according to number of *concepts* represented, quantity of valid *relationships* between nodes, the levels of *hierarchy* in the structure, *examples*, and *cross-links* (Novak & Gowin, 1984; Wallace & Mintzes, 1990) according to the following criteria.

concepts: one point each
relationships: one point for each valid proposition
hierarchy: five points per valid level
examples: one point for each valid example
cross-links: varied from one to five points each, depending on complexity

Scores in the five categories for each map were recorded, as well as totals for each map. Correlations between the total scores on the two concept maps were derived, as well as correlations between the mathematical concept map score and number of units mastered in the course.

Interviews

The interview questions were scored by the researcher and a mathematics teacher unfamiliar with the students or the construction of concept maps. The arithmetic and elementary algebra interviews were heard by two different teachers. The teacher listened to the tape of the interview and examined the work completed by the student during the interview. Although part 3 of the interview was included on the tape, the teachers did not listen to that part of the interview, being unfamiliar with the techniques of map construction. The teacher then separated students into two groups, according to how well the student connected ideas in mathematics during the interview. These results were compared with the researcher's scores on the same interviews, and with the original group selection of students for the interview.

Results and Discussion

Comparisons between the data collected for each class indicate that the student generated concept map representations produced conclusions about the student's complexity of knowledge connections similar to the data obtained from the interviews. There was 83.3% and 91.7% agreement between the teachers' placement from the interview and the initial placement by concept map scores. This corresponded to the researcher's placement of students into groups with 91.7% and 83.3% agreement between the researcher and the two mathematics teachers. The disparities occurred with students who were borderline with the scores of the middle group in the initial placement according to the mathematical concept map.

The scores of the concept maps also were significant when compared with student progression in the course, with comparisons in the arithmetic course ($F(1, 70) = 33.32$, $p < 0.0001$) being more highly significant than those in the elementary algebra course ($F(1, 44) = 7.18$, $p < 0.05$). Comparisons between the two concept map scores in each class showed only the results from the arithmetic students being even marginally significant ($F(1, 70) = 3.45$, $p < 0.10$)

Conclusions

This study was designed to examine the use of concept maps as a research tool in mathematics education. We originally proposed to examine their use in two distinct avenues: correspondence between concept maps and clinical interviews in determining concept connectivity, and ability of concept maps to predict academic achievement. The results of this study would indicate that student generated concept maps show indications of validity as a research tool, giving comparable results to both interviews concerning student knowledge of relationships between concepts and academic progression of students through an individually-paced course.

The connection between concept maps and clinical interviews appears quite promising as a research tool. Concept maps have an tremendous advantage over interviews in the shorter time required to execute and analyze. They can be administered several times over the course of a study, with no additional researcher-time required to instruct subjects in their construction. Comparisons between mathematical concept maps and those derived from another area by the subjects indicate that ability to construct concept maps alone does not explain differences between scores obtained by subjects on the maps. We contend that the concept maps do indeed tap into many aspects of concept connection that heretofore have been accessible only by way of the clinical interview.

However, one should be careful not to extend these noted results to academic achievement in more traditionally, group-taught mathematics courses, without further research. In order to master a large number of units in the courses of this study, connections with previously covered units were required, a condition not normally observed in most mathematics classrooms.

Further studies need to be conducted before concept maps can be accepted as valid substitutes for clinical interviews concerning concept connections in mathematics. These studies should include work with similar populations in a more traditional type of math course, as well as extensions to additional populations. Improvements could also be made in the analytical framework of the interview used in this study by a more rigorous detail to the form, as well as the quantity, or concepts used in the interview. Examination of the various ways that concept maps have been used in science education will hopefully open the door to further research concerning concept maps in mathematics education, as well.

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COLLABORATIVE ACTION RESEARCH IN MATHEMATICS EDUCATION: A TALE OF TWO TEACHER-RESEARCHERS

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First steps toward engaging in collaborative action research in the mathematics classroom are vital in establishing quality research projects designed and implemented jointly by classroom teachers and university faculty. This brief paper describes the initial phases of a collaborative research project centered around the incorporation of problem solving into a middle school mathematics curriculum. Brad, a seventh-grade mathematics teacher, and I established a collaborative relationship and subsequently developed, implemented and documented our efforts regarding mathematics classroom reform. Described herein are details of the beginning stages of our collaborative research project.

Classroom inquiry conducted by mathematics educators has taught us much about the dynamics of teaching and learning mathematics and has resulted in suggestions for reform in mathematics education (e.g. National Council of Teachers of Mathematics [NCTM], 1989). However, research that is initiated by and documented by classroom teachers has the potential to result in immediate classroom reform. According to Miller and Pine (1990), "Teachers can contribute to educational improvement by conducting classroom research concerning teaching and learning processes. This "action research" role for teachers can enhance the professional status of teaching, generate theory and knowledge, increase the effectiveness of improvement efforts, and promote teacher development (p. 57)."

What Is Action Research?

Miller and Pine (1990) define action research as "an ongoing process of systematic study in which teachers examine their own teaching and students' learning through descriptive reporting, purposeful conversation, collegial sharing, and critical reflection for the purpose of improving classroom practice (p. 57)." Clift, Veal, Johnson, and Holland (1998) suggest that collaborative action research is characterized by its focus on practical problems of individual teachers or schools and its emphasis on professional development and support for collaboration between teachers and university staff. Although the notion of valuing "teacher as researcher in the classroom" is growing among educational researchers (Cardelle-Elawar, 1993; Rafferty, in press), probably the most difficult phase of conducting action research is encouraging teachers to believe in their abilities to be effective classroom researchers. Thus, collaboration between the university researcher and classroom teacher is a vital first step in the process.

As a member of a collaborative inquiry team studying action research, which is a subgroup of a larger consortium, Project UNITE (Urban Network for the Improvement of Teacher Education), I have made efforts to become involved in several action research projects with classroom teachers at elementary and middle schools in Indiana. The focus of this paper is to describe one particular collaborative action research project that is in its early stages of implementation. Herein I discuss how the collaborative relationship was established and how the research project itself was initiated and designed.

Establishing A Collaborative Partnership

The key to finding an "ideal" classroom teacher with whom to engage in action research is to let it happen naturally. I recognized this after having attempted to initiate collaboration with several teachers who were either not interested in personal classroom research or not at a point where they wanted to commit to such a project.

The successful collaborative relationship about which this paper is written emerged as a result of discovering that a classroom teacher enrolled in one of my graduate courses was interested in mathematics classroom reform. While teaching the graduate course, in which the emphasis was "teaching mathematics through problem solving," I engaged the students (classroom teachers) in problem solving and demonstrated the teacher's role in involving students in such tasks. Together, the teachers and I discussed how one might implement problem solving into an existing curriculum, talking specifically about what issues need to be considered. Throughout the course, one middle school mathematics teacher, Brad, expressed a strong desire to improve his own problem-solving abilities as well as to augment the quantity and quality of problem solving in his seventh-grade classroom. After several conversations, I discovered that Brad was excited about making problem solving a focal point of his classes, but was unsure about the best way to make changes in his current teaching. With no thoughts in mind about collaborative research, I volunteered to come into Brad's classroom on occasion and "co-teach" problem solving lessons if he felt that would help with implementing changes in his classroom. Brad approved of the idea of co-teaching.

As the graduate course progressed, I realized that helping Brad to document his attempt to incorporate problem solving into his existing curriculum was an opportunity for collaborative research. Consequently, I told Brad about my interest in learning more about collaborative research and asked him if he would be interested in documenting his progress over the next school year and eventually share his findings with other mathematics educators. He responded positively, emphasizing that he would like to learn how to conduct classroom research. In addition, the possibility of reporting findings about the successes and limitations of his attempts to teach problem solving in the "spirit of the NCTM *Standards* (1989)" was very appealing to him. Overall, Brad was comfortable with making this a team effort. Thus, our collaborative partnership for researching mathematics curriculum reform was established.

Brad's Initial Reflections on Problem Solving

To begin the research process, Brad and I discussed the role of reflective writing in qualitative research. We both agreed to keep journals which we would share with each other regularly. I suggested some initial questions for him to consider and reflect upon before beginning the study. The first writing prompt I provided for Brad was to write about his "relationship with problem solving." This included addressing the following questions: (1) Why do you think problem solving is important?, (2) What experiences with problem solving did you encounter as a student? (3) Do you enjoy engaging in problem solving? (4) Do you feel successful as a problem solver? (5) In what ways do you currently involve your own students in problem solving? (6) In what ways would you like to improve your abilities to solve problems and to teach problem solving in your classroom? Brad began his reflection by expressing his views on the value of problem solving.

> *I have come to realize that all routine mathematics exist for one purpose - problem solving. Knowing how to solve manipulations does not mean we have solved the problem. . . Problem solving promotes an understanding of mathematics as a whole. . . . I do believe that problem solving is a much needed skill that must be a priority in our curriculum if our students are to be successful in mathematics and their world.*

In addition to describing his views on the benefits of problem solving, Brad summarized his experiences with problem solving as a student, explaining that those experiences were not personally motivating.

> *As a student myself I do remember enjoying mathematics in the elementary years. As I progressed to middle school and high school that excitement faded. . . . Problem solving was presented in just one chapter and I can remember dreading it all year. . . Like a child who holds his breath while eating something he dislikes, I found myself developing an attitude of doing whatever it took to swallow this bitter pill and survive.*

When asked to reflect on the problem solving as it is currently incorporated in his own classroom, Brad explained that although he does include problem solving in his curriculum, he does not have his students pursue problem solving to the extent that he had been challenged to do so in his graduate course.

> *Occasionally I will start class with a written story problem which students are asked to solve themselves or in a group. I have at times given students situations which require research outside of class and then asked them to apply their findings to solve their problem. . . As a teacher of mathematics, I have much room for improvement . . . My hope is to present [problem solving] in a way which stimulates the student's desire to continue [to learn mathematics].*

Planning For Implementation

In addition to reflecting on the role and/or value of problem solving, I encouraged Brad to identify what he considered to be the key issues to address when establishing problem solving as a focal point of his mathematics classroom. Brad identified five factors to consider when planning for the implementation phase: (1) time, (2) evaluation, (3) instructional material and training, (4) problem solving strategies, and (5) use of manipulatives. He elaborated on each of these elements in a reflection on planning for classroom reform.

Time

Regarding the issue of time, Brad explained:

> *Time will be a significant factor as I implement problem solving into my classroom. Ideally I would like to spend at least 20-25 minutes twice a week solving process problems. If done on a consistent basis, I believe this will be enough time to adequately cover problem solving. My only concern is that I cover the material my seventh graders will need to be successful in the eight grade.*

Evaluation

Brad was concerned about how problem solving should be evaluated, especially during this "pilot" year of implementation.

> *Evaluating a students' problem solving performance and progress will indeed be a difficult task. I do believe there must be evaluation and that students must be held accountable for their effort, attitude, and learning of problem solving strategies. Because many students are easily discouraged by non-routine mathematics I must be careful in the area of evaluation. I prefer to assess problem solving holistically rather than using a detailed problem-solving assessment tool. I will put more emphasis on the student showing a correct problem-solving strategy than deriving the correct answer. Although I have an idea of what I believe is important in the area of problem solving evaluation, I cannot say precisely how it will be at this time.*

Instructional Material and Training

Brad recognized the benefits of engaging in collaborative research with a university researcher.

> *For some time I have wanted to implement problem solving into my classroom curriculum. After sharing this desire with Anne [author/co-researcher] We decided to collaborate in establishing a problem-solving program in my classroom. . . . This arrangement will allow me to utilize her expertise and resources while in a classroom setting.*

Problem-Solving Strategies

When writing about problem-solving strategies in his journal, Brad stressed that he planned to spend a considerable amount of time with his students discussing and reflecting on characteristics of good problem solvers. He listed ten specific problem solving strategies that he planned to incorporate into his program. They include: (1) trial and error, (2) draw a picture, (3) make an organized list, (4) make a table, (5) work backwards, (6) look for a pattern, (7) use logical reasoning, (8) use simpler numbers, (9) try a smaller case, and (10) Venn diagramming.

Use of Manipulatives

Brad remarked that manipulative resources at his school are scarce. One of his goals is to increase the quantity and variety of manipulatives for use in his classroom.

> *Research has shown that the use of manipulatives with students in the middle school greatly increases their understanding of the concept being taught. Such experience with "concrete" materials helps them connect their understanding to their world. Whenever possible, manipulatives will be used [in my classroom] to assist students in their understanding and implementation of problem-solving strategies.*

Documenting Classroom Reform

Having been guided to engage in some initial reflections about the kinds of changes he might want to employ, Brad began to get a sense of how the research phase might unfold. Although I offered some suggestions of how we might want to proceed with the study, I expected Brad to make the final decisions. He determined that I should come into his classroom one day each week to co-teach a problem-solving activity with his first-period class. He would then implement the same lesson to subsequent classes.

To document the progress of students' learning and curriculum reform, I suggested that Brad maintain a detailed "notebook" which would include a curriculum outline, samples of students' work, and his own notes on or reflections about the project. Brad elected to do this and decided that following each problem-solving lesson, he would collect and store samples of his students' work in the notebook. He agreed that it would be beneficial to reflect on the lesson, addressing issues such as (1) what went well? (2) what would he change? (3) how did students perform? and (4) in what ways are students' problem-solving abilities changing? It was determined that I would also reflect on my teaching in his classroom as well as my view of the students' growth as problem solvers. In addition, Brad and I would meet monthly after school to share our reflections, to make any necessary curricular revisions, and to analyze the documentation of improvement in both the students' problem-solving abilities and our own teaching methods. We also planned to tape record our discussions ,and later reflect individually on our meetings.

Summary

We were now ready to embark on our data collection phase. The excitement that Brad demonstrated regarding the partnership, as well as the entire "curriculum reform-research project," was tremendous. It was clear that Brad gained confidence in his ability to "investigate" in his own classroom. Eventually, my role as the university investigator in the collaboration became that of the "debriefer" as described by Lincoln and Guba (1985). After being guided to reflect on his role as both teacher and researcher, Brad expressed an eagerness to continue to engage in and report on action research in his own classroom.

Upon engaging in collaborative research, I found that I played multiple roles. Primarily, I taught Brad about planning for, documenting, and reporting about research. However, in this particular study I also had the opportunity to share my knowledge of and experience in teaching problem solving and implementing curriculum reform by demonstrating these activities in his classroom. In addition, I learned that a university researcher's involvement in action research is very different from the more directive role often taken in other research projects. This joint effort has taught me much about action research, and has, I believe, resulted in true classroom reform. I am convinced that when action research occurs, the results are immediately applicable to the classroom. At the same time the benefits, in terms of what both classroom and university teacher-researchers learn regarding collaborative research, are long-lasting.

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Social and Cultural Factors

RESEARCH PAPERS

Middle School Students' Perceptions Of Their Everyday Mathematics Usage 77
Joanna O. Masingila

A Basis For Equity In Mathematics Education: An Experiment In Cultural Course Development 84
Norma C. Presmeg

A Case Of Equity Reform In Mathematics: The Students And Their Response 91
B. F. Risacher

The Negotiation Of Social Norms In A Mathematics Class 98
Sandra Davis Trowell and Grayson H. Wheatley

Exploring The Social In Social Construction 105
Steven R. Williams and Kathy M. C. Ivey

MIDDLE SCHOOL STUDENTS' PERCEPTIONS OF THEIR EVERYDAY MATHEMATICS USAGE

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This study examined middle school students' perceptions of how they use mathematics outside the classroom in an attempt to learn more about and build on students' everyday mathematics practice and close the gap between doing mathematics in school and out of school. Twenty middle school students were interviewed before and after keeping a log for a week where they recorded their everyday mathematics usage. Through the interviews and log sheets, I found that the mathematics that the middle school students perceived they used outside the classroom could be classified as one of the six activities that Bishop (1988) has called the six fundamental mathematical activities.

Introduction

Students need in-school mathematical experiences to build on and formalize their mathematical knowledge gained in out-of-school situations. An important part of a mathematical experience in school is the guidance and structure that can be provided by a teacher to help students make connections among mathematical ideas. By building upon the mathematical knowledge students bring to school from their everyday experiences, teachers can encourage students: (a) to make connections between in-school and out-of-school mathematics in a manner that will help formalize the students' informal mathematical knowledge, and (b) learn mathematics in a more meaningful, relevant way. "Mathematics teaching can be more effective and will yield more equal opportunities, provided it starts from and feeds on the cultural knowledge or cognitive background" of the students (Pinxten, 1989, p. 28).

However, in order to help student connect doing mathematics in school and doing mathematics out of school, we need to know how students use and perceive how they use mathematics in everyday situations. To that end, the research study discussed here was conducted to gain some insight into how middle school students perceive that they use mathematics in out-of-school situations.

Methods and Data Sources

During the first phase of the study, 20 middle school students were interviewed to determine their perceptions of their everyday mathematics practice. Ten eighth-grade students and ten sixth-grade students participated as respondents. Five of the eighth-grade students and five of the sixth-grade students were from an urban school and the ten other students were from a suburban middle school. The respondents were asked questions such as, "How do you use mathematics outside the mathematics classroom?" "Describe a situation where you use mathematics outside the mathematics classroom." "What do you think mathematics is?"

Following the interviews, the 20 respondents and other volunteers from their classes kept a log for a week to record their use of mathematics outside the mathematics classroom. A total of 69 sixth-graders and eighth-graders kept logs for a week. The directions were simply to describe how and where they used mathematics. Students were given a new log sheet each day to record their observations for that day.

After the students kept logs for a week, I analyzed the logs and categorized their mathematics usage. I also analyzed the interview comments, looking for categories of mathematics usage. I then interviewed the 20 respondents again, asking them to: (a) clarify things they had written in their logs, (b) expand on earlier responses, and (c) discuss any insights they had after keeping a log for a week. Throughout the study, I was looking for categories of mathematics usage in out-of-school situations, changes in the students' perceptions after keeping a log, and any evidence of students making connections between their school mathematics and out-of-school mathematics.

Findings

Through the interviews and log sheets, I found evidence that the mathematics that the middle school students perceived they used outside the classroom can be classified as one of the six activities that Bishop (1988) has called the six fundamental mathematical activities. Furthermore, I found some evidence of all six activities. These activities, Bishop argued, are "both universal, in that they appear to be carried out by every cultural group ever studied, and also necessary and sufficient for the development of mathematical knowledge" (p. 182). The six activities are counting, locating, measuring, designing, playing, and explaining.

Counting is "the use of a systematic way to compare and order discrete phenomena. It may involve tallying, or using objects or string to record, or special number words or names" (Bishop, 1988, p. 182). Examples of data reported by the middle school students that I classified as counting include: counting change in a store, estimating the prices of several items, figuring mileage for a trip, finding someone's age given her or his year of birth, calculating the amount of ingredients needed when changing a recipe to serve a different number of people, keeping score during a basketball game, comparing unit prices, counting the number of soda cans and figuring amount of money to be received, figuring tip in a restaurant, counting music measures in band.

Locating is "exploring one's spatial environment and conceptualising and symbolising that environment, with models, diagrams, drawings, words or other means" (Bishop, 1988, p. 182). Data reported by the middle school students that I classified as locating include: using a map to plan a driving route, working with a parent to use a blueprint to determine where to install some electrical wires, drawing a sketch before building a club house.

Measuring is "quantifying qualities for the purpose of comparison and ordering, using objects or tokens as measuring devices with associated units or 'measure-words'" (Bishop, 1988, pp. 182-183). Data I classified as measuring include: cutting material before sewing a shirt, estimating distance on baseball field, estimating size of hole needed for a screw, measuring parts for a rocket, comparing skating speed in street hockey with and without ball, measuring ingredients before cooking, measuring picture to determine frame size, measuring food portions, laying out measurements for club house, figuring area of garden, measuring pulse rate during track practice, helping parent measure house for siding, figuring spacing for outdoor ground lights.

Designing is "creating a shape or design for an object or for any part of one's spatial environment. It may involve making the object, as a 'mental template,' or symbolising it in some conventionalised way" (Bishop, 1988, p. 183). Examples of data I classified as designing include: designing a patio, designing a club house, carving designs out of wood, planning a flower garden, designing a rocket, creating a video project.

Playing is "devising, and engaging in, games and pastimes, with more or less formalised rules that all players abide by" (Bishop, 1988, p. 183). Data reported by the students that I classified as playing include: playing chess and considering different strategies, playing lacrosse and figuring best position to run to, playing board games and figuring what moves to make, playing Perfection and determining where to put all the pieces, playing baseball and figuring where to run to catch the ball, playing basketball, playing video games, playing Number Scrabble.

Explaining is "finding ways to account for the existence of phenomena, be they religious, animistic or scientific" (Bishop, 1988, p. 183). Data I classified as explaining include: using mathematics to figure things out in science, explaining why a model train set on the same speed goes faster on one incline than on a steeper incline, explaining why a pan of food on a larger burner heats faster than one on a smaller burner, explaining why a person will feel colder at one temperature with a certain wind speed than at a lower temperature with a lower wind speed.

During the first interviews, students generally reported two to three examples of how they perceived themselves using mathematics outside the classroom.

The vast majority of the data came from the logs, with students recording from three to nine examples each day of how they saw themselves using mathematics. Relatively few new examples were reported during the second interview as most of the questions centered around clarifying what the students wrote in their logs.

In general, the students' ideas of what is mathematics did not change from the first interview to the second. Examples of student responses to this question include: "Mathematics is a subject that helps me a lot." "Mathematics is a system that you use to figure things out." "Mathematics is a way of looking at things—collecting data and using what you know to find out other things." "Mathematics is using numbers and principles to find solutions."

Although students' ideas of what is mathematics did not change, all but one of the 20 students thought that they learned from keeping the log and that it was beneficial: "I noticed that I used it [mathematics] a lot more than I thought; doing the log made me concentrate and I noticed a lot more things." "It [keeping a log] opened me up to see how much more I use it [mathematics] than I thought I did. It made me think, hey, I'm using math. Its [mathematics] is just an everyday thing and its more important than I thought." "I'd fill out my log in the evening and think back over everything I did and how mathematics was involved." "When I did my log I would sit on my porch and watch people go by and think about the people and how they use math." "[The logs] helped me think more about the subtle and less obvious places where you use math."

Discussion

Because I examined student-generated data, any analysis of the data has to consider that these are the students' perceptions of how they use mathematics outside the classroom. The gap might be very wide between their perceptions of how they use mathematics and how someone might perceive them using mathematics if that person observed them in their everyday activities since one's perception of how one uses mathematics is a function of what one thinks is mathematics.

My analysis is that the primary reason for this gap is that the students' perceptions of mathematics are generally their perceptions of school mathematics. In other words, when the students looked for examples of how they used mathematics in their everyday life, they were in essence looking for examples of how they used school mathematics in their everyday life. By way of example, I noticed that during the interviews, when the students were trying to answer the question about how they used mathematics outside the classroom, they often verbalized that they were trying to think of situations that involved numbers. Several students mentioned that they think mathematics "is the study of numbers." Thus, it was quite natural for them to think that numbers must be involved in some way in their mathematics usage outside the classroom.

Some students appeared to understand that there is more to mathematics than school mathematics. One student mentioned that he used mathematics and mathematical thinking a lot but he could not articulate any specific examples that were different than ones involving school mathematics: "Its different than what we do in school—I can't explain it—but I know it is mathematics." Another student mentioned that "the logs helped me realize that I did not use math that much outside of school." It may be that she was not aware of when she used mathematics; another interpretation might be that what she saw herself doing outside of school did not resemble school mathematics and so she did not classify her activities as mathematical.

Conclusion

This research is one step in the direction of closing the gap between doing mathematics in and out of school. If we can understand better how students perceive that they use mathematics outside the mathematics classroom, we can build upon that mathematical knowledge and extend and formalize it in the classroom. Talking with students and having them keep a log of their mathematics usage is one way for teachers and students to learn about how perception of what is mathematics may help students to change their perceptions of how they use mathematics outside the classroom.

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A BASIS FOR EQUITY IN MATHEMATICS EDUCATION: AN EXPERIMENT IN CULTURAL COURSE DEVELOPMENT

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The research question addressed in this report is whether a theoretical model developed on the basis of literature on equity and culture in mathematics education is viable in the development of an ethnomathematics course for prospective and practising teachers of mathematics. According to qualitative data collected from students in the course and in participant observation, this theoretical model was judged to be eminently viable for this purpose.

RATIONALE

The underrepresentation of some minorities in higher level mathematics classes has been amply documented (Mathematical Sciences Education Board, 1990; Secada, 1992). With the concern that mathematics education should become more equitable, the problem arises of how to achieve this goal. The issues are of more than local or even national interest; increasingly heterogeneous school populations in many countries require that educators need to rethink traditional pedagogy, curricula, and structuring of schools (Nieto, 1992). The issues are well known, but in mathematics education a research base grounded in theory and in practice is just beginning to be established. This paper represents a step in this direction.

A THEORETICAL MODEL

The first principle in this model is that diversity in a mathematics classroom is not a liability but an assett. Diverse cultural backgrounds may be regarded as a rich resource for teachers (Nieto, 1992; Bishop, 1988). This position is an antidote to the 'color blindness' often held in good faith by teachers and whole school districts as a recipe for equity, which may in fact have the opposite effect since it results in the hegemony of the dominant culture which disadvantages all others to a greater or lesser extent (Nieto, 1992). This principle of culture as a resource resonates well with the "Connections" standard which is advocated for all grades, K - 12, in the NCTM's (1989) *Curriculum and Evaluation Standards for School Mathematics*. However, it is a principle which does not yield set packages of 'multicultural materials' to be used in cookbook fashion, although some multicultural materials and activities are valuable as a starting point.

This reasoning leads to a second principle, namely, that every student is regarded as having ethnicity. The principle is a state of mind which affirms diversity, is sensitive to the needs of individual students, and thus adapts pedagogy in a variety of ways to take into account and use the ethnic backgrounds of all students. Thus the principle involves the epistemologies, beliefs and attitudes of teachers with regard to multicultural education, and even their ontologies, since questions about what mathematics is, and where it exists, inevitably arise. For instance, one of the questions which it was found necessary to examine at several points in the course, was as follows: "If carpenters and seamstresses are using mathematical principles without being aware of them or calling them mathematics, are they in fact doing mathematics?" (Millroy, 1992; Fasheh, 1989).

The third principle in this model is that mathematics is a cultural product. Bishop's (1988) book provided structure and a theoretical framework for putting this principle into practice. Six "environmental activities", namely, measuring, designing, counting, locating, playing and explaining, are found in all cultures (ibid.). These activities are the basis for the development of mathematical concepts. Bishop's framework allows for five "significant levels of scale" in the social aspects of mathematics education, i.e., cultural, societal, institutional, pedagogical and individual (p. 14). The constructs of environmental activities and levels of scale enabled course participants to situate their theoretical papers and presentations in a larger framework which facilitated subsequent analysis and disciplined inquiry. All six of the environmental activities were represented in the topics chosen by the students, and to a greater or lesser extent in individual topics, all five levels of scale were involved. Informed also by Bishop's analysis of values involved in mathematics education, these constructs provided a theoretical framework for culturally based classroom activities from which mathematics was developed in an environment which affirmed the cultural values, practices and beliefs of diverse students.

COURSE DEVELOPMENT

In an experimental graduate level course for prospective and practising teachers, "Mathematics Education and Culture", these principles were put to the test. Resonating with the qualitative nature of this research in curriculum development, a preliminary course outline was flexible to allow progressive focussing as the three principles were implemented. 21 adult students from a variety of ethnic backgrounds enrolled in the course. According to the second principle, that every student has ethnicity, each student was required to explore this ethnicity for the dual purposes of writing a theoretical paper, within Bishop's (1988) framework, and of involving the graduate class in cultural activities which facilitated the construction of mathematical concepts. Some of the practising teachers were also able to try out these activities with their own students, at elementary, middle and high school levels. After the first few weeks of discussion of theoretical issues and of examples introduced as practical activities by the instructor (the writer, acting as teacher-researcher), each student was able to identify one or more environmental activities from his or her cultural background for the theoretical and practical requirements of the course. The choice of topic was in some cases difficult. Data from journals and interviews indicated that particularly the African American students had to grapple with the belief that they had "lost" their culture (Holloway, 1990). Some students made compromises, choosing an African topic (e.g., Egyptian arithmetic) for their theoretical paper and an American activity (e.g., designing an entertainment center) for the practical component. Topics chosen included the mathematics of an African American sharecropper, of baseball and marching bands, of German and African folk songs, of the counting system used in signing by deaf Americans, of the South Korean flag, the Chinese abacus, and of traditional navigation between the islands of Micronesia, as well as of cultural games played on several continents.

No more than a couple of the activities based on these topics can be described here. Jenny had attended school with deaf students and could sign fluently. After discussion, the class agreed that deaf culture was a viable candidate for environmental activities. Jenny's theoretical paper and the first part of her presentation introduced the class to the history of schools for the deaf, going back to the Abbe de l'Eppe in France in 1755. Jenny's talk was dramatized by her use of American Sign Language as she spoke. Then she asked the class ot break into small groups and collaboratively to solve the problem of how one might represent the numbers from one to 999 using the fingers of one hand. After each group shared a solution with the class, Jenny taught the "official" way of signing these numbers. Later, students expressed that they had learned far more than merely how deaf people communicate numbers. The activity of trying to work this out for themselves had caused them to reflect on the "chunking" properties of the base ten system of numeration, how and why it worked. (The environmental activity used here was counting.)

A second example, which was introduced by the instructor (who is from South Africa), is the artwork with which the Ndebele people of South Africa decorate their house exteriors. Pictures of Ndebele houses were distributed for students to analyze in groups. After geometric figures and their complex interrelationships in the designs were identified, each student attempted to create his or her own "Ndebele" design using geometric figures and arrangements like those analyzed. (The environmental activity used here was designing.)

These two activities were judged to be suitable for developing mathematical concepts in elementary or middle school. However, some of the activities which students chose introduced high school mathematics, e.g., Phillip's development of mathematics involved in activities of marching bands introduced exponential equations.

SOME CONCLUSIONS

This course was regarded by the participants as a starting point in developing multicultural awareness in mathematics education. For equity, such awareness is requisite in mathematics teachers at all levels (Nieto, 1992). The positive reactions of these students to their own cultural mathematical learning and reflecting, and of students in their school classes in some cases, suggest that multicultural mathematics is "an idea whose time has come" (Beverley J. Anderson, MSEB, 1990, p. 3). It remains to be seen, and further research would be required to investigate, whether schoolchildren could develop mathematical constructs from their cultural activities as successfully as did these adult students from their ethnic backgrounds, taking ownership of the course in the process. But the present study does suggest that the theoretical principles used in this model are viable for cultural curriculum development in graduate courses for teachers. Such courses could be an important starting point in the development of a basis for equity in mathematics education.

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A Case of Equity Reform in Mathematics: The Students and Their Response

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This work comes from an equity reform project at a large inner-city high school which attempts to have all students succeed in two years of college preparatory mathematics. The project has many areas of emphasis and is supported by several industry and academic "partners." This paper concentrates on describing the beliefs and characteristics of the Algebra I students and the progress of Hispanic and African Americans during the first year of the program. Results indicate the students value math and want to attend college, but know little about college requirements. The success of the target students in Algebra was very low and few used available help services. Attendance, language difficulties , beliefs , and attitude appear to be factors.

Background and Objectives

Area of Concern

"There is no greater educational problem in our nation than improving the educational opportunities and outcomes for students who are placed at risk of educational failure," (Winter 1994 OERI Bulletin). This concern is reflected by The Curriculum and Evaluation Standards for School Mathematics (1989), which recommends that all students study a core curriculum in mathematics that includes algebra and geometry sufficient to allow them the option of pursuing higher education. While the mathematics educational community endorses the importance for all students to obtain "mathematical power," there is a great deal of inequity in the achievement in mathematics by ethnic groups (Dossey, et al., 1988; Howe, D.K. 1994). The under representation of non-Asian minorities in all scientific and engineering fields is widely cited as a result of their insufficient mathematics preparation (Everybody Counts, 1989). Algebra I has a particularly high failure rate and appears to be a filter to keep many students out of higher mathematics and limiting their access to higher education.

The theoretical assumptions of this work are consistent with current thinking on equity and access to mathematics for all students. This philosophy asserts that all students are capable of reasoning mathematically and succeeding in a college preparatory curriculum. This study also reflects the theories of systemic reform which indicate that making a significant change in a mathematics program requires broad changes in many areas, i.e., the curriculum, the teachers, the students, the parents, and the school administration. In contrast to a systemic approach, some schools are simply requiring all students to take traditional Algebra, and "'Algebra for everyone' has been a dream that for some has turned into a nightmare," with higher levels of failure for the at risk students (Price, 1994).

Objectives of this Paper

This paper examines a broad based effort at an inner city high school to increase the success of underrepresented minorities in college preparatory mathematics. The reform involves many areas of the learning environment and is supported by a number of different partnerships. While all of the components of the project are an integral part of the reform, an examination of the student population, their beliefs, feelings and reactions to the reform is of primary concern and is the focus of this paper. Consequently, this paper will only briefly describe the program in order to concentrate on the students and their progress during the first year of implementation of the reform. It is hoped that this analysis of the students will provide useful information and encourage additional research and reform in this crucial area.

General Description of the Project

The high school of this study is located in a large west-coast city within an area of urban sprawl where minorities are the majority. The school has approximately 2,297 students with the large majority being Hispanic and Southeast Asians and Pacific Islanders. . Historically, few students at this school pursue college preparatory mathematics; for example, only 24% of the Hispanic and African American students were enrolled in or succeeding in college preparatory mathematics courses during the year prior to this study.

The reform involves a group of "partners" exerting a cooperative effort to enable all students to succeed in college preparatory mathematics. The partners are from the mathematics department of the high school, a local junior college, a large nearby state university, and representatives from several local high-tech industries. The decision was made to require all incoming ninth grade students to take Algebra I and subsequently, Geometry, rather than allow them to choose general math courses. The specific goal is to reform and support the mathematics program in a variety of ways to enable ninth grade Hispanic and African Americans to successfully complete Algebra I and Geometry within three years.

Areas of Special Emphasis and Support

Faculty Support. Support for the teachers include summer equity workshops, whole day planning retreats, and teacher team meetings several times per month. Several teachers had reduced teaching loads during the year to work on the project, and the administration attends some project planning meetings and contributes secretarial support to the project. .

Student Support. Support available for students included additional tutoring, both in-class and every day after school by industry and college student volunteers and by peers tutors. Students had additional counseling and some were mentored by industry volunteers. Students had extensive summer school opportunities and encouragement to take an "Algebra Booster" or to repeat Algebra or Geometry if they received a D or F in the course.

Family Support. Support for family involvement with the students and mathematics included "Back To School Night," two "Family Math Nights," and several school presentations concerning cultural awareness in mathematics. There was also an evening class offered in "Algebra for Parents," and teachers increased their efforts to contact parents by telephone calls and letters.

Curriculum Support. The curriculum was changed from a traditional text to an innovative problem-solving type of text with recommended instructional techniques of small group work, use of computers, graphing calculators, portfolios, and open ended assessment tasks. Students had increased opportunities to use graphic calculators and computers within the classroom as part of Algebra I.

Methodology

Written Surveys

All of the Algebra I students (grades 9-12) responded to an initial written survey of 41 questions concerning their attitudes and beliefs about mathematics and college requirements, family support for education, and language spoken at home. These were scored by an external testing agency. Two surveys were administered during the year on (1) the students' feelings about the course and textbook units and (2) the teachers' feelings about the program and their students' progress.

Personal Interviews

Personal interviews were conducted by a mathematics education researcher at the end of Algebra I with twelve ninth grade students. These students were volunteers and represented the range of achievement levels and half were from the target group. Field notes and audio recordings were made of the interviews.

Academic Records and Special Services Records

Mathematics achievement of the ninth grade target students was assessed using eight grade standardized mathematics test scores, the semester and final grades in Algebra I and by replies to open-ended problems.

Other factors of interest were documented, such as, attendance, grades in other subjects, and number of in-class and after school tutoring sessions. Comment cards were requested from tutors, counselors, etc. who provided "special services" for small groups or individual students. Statistical analyses were used to describe the students and their progress.

Results

Initial Fall Survey of All Algebra I Students

Makeup of Group. Responses from the 640 Algebra I students on the fall survey reflected the following ethnicity's: 52% Mexican-American or Latino, 7% African American, 27% Asian Americans, 4% Filipino, 4% While, non-Latino, 6% other groups (< 2% each). This group was 53% males and 47% females, with the following grade levels: 45% in 9th grade, 26% in 10th grade, 16% in 11th grade, and 11% in 12th grade. Forty nine percent of the students reported that English is not the language spoken at home. At the teachers' request, the fall survey was also printed in English, Spanish and Vietnamese due to the concern that many students would not be able to answer the survey in English.

Family Background. Most of these students had family backgrounds with little emphasis on education, i.e., many students reported that their parents had probably not finished high school (60%). However, almost all students indicated they want to finish high school (93%) and most are interested in going to college (81%). Students reported that most of the parents also want them to attend college (83%). However, about a third of the students replied they knew little about the courses needed for college or much about different types of colleges.

Beliefs and Feelings about Math. These students expressed interest in doing well in school and in mathematics (86%). They believe one needs to "know math to get a job that pays good money (79%)," and that doing well in math is important to go to college (83%). In contrast to these beliefs, only 5% reported attending a summer "bridge to algebra" program available to all students as a booster to do well in algebra. The majority of students said they liked math (63%), but only 24% said they would be interested in belonging to a math club. About a third said that "Math is boring." Many students appeared to doubt their ability to do math and replied they "can handle other subjects okay, but I mess up in math" (53%).

Individual Interviews at the End of Algebra I

View of Mathematics. The individual interviews with 12 ninth graders at the end of Algebra I showed that only about a third defined "mathematics" as a method or a way to solve problems. The majority gave simplistic responses such as "it's a course in school you have to take." About half describe a "typical math problem" as an arithmetic operation on numbers rather than any sort of situational problem or algebraic equation. Almost all felt that they took between 1 and 10 minutes to do a math problem, provided they already knew how to do it. Their idea of doing math problems was reproducing a known procedure.

Feelings and Beliefs About Math. Many compared doing math to facing a tremendous and overwhelming challenge. Most said they liked mathematics, but the majority used "frustrated" and "confused" as the most common words to describe their feelings in math class. About a third felt that some people are just naturally good at math and another third said that practice makes you good at math. Students claimed that working in groups is what makes doing math fun in class and they work with brothers or sisters and friends outside of class.

Value of Math and Education. All of the students felt they would need math when out of high school, but about a third said they do not use math outside of math class now. All twelve students expressed intentions to go to college and cited career goals such as doctor, lawyer, engineer, and scientist.

Competence with Linear Functions. Only one third of the students could graph two linear equations which were already in slope-intercept form, and only one fourth could solve the system algebraically. None of the students could solve a simple word problem involving number of items and cost per item. No student even tried to set up two equations with two variables. Most did not appear to understand the relationship that quantity x price = total cost. Some estimated an answer but did not check the estimate in the problem situation to see if it worked.

Progress of Hispanics and African Americans In Algebra I

Math Achievement Scores. There were 151 Hispanic and 23 African American students in Algebra I. According to their 8th grade standardized testing, these students scored generally around the 45th percentile in Math Computation, Math Problem Solving and in Combined Math Score, with female scores slightly higher than males. The final averages in Algebra I for both sexes were around 62% or a D- letter grade, with a C- required to advance to Geometry. The distribution of grades was 6% A's, 6% B's, 11% C's, 19% D's, and 58% F's. It should be noted that the 8th grade standardized test data was a fairly normal distribution and is definitely not correlated to the grades in Algebra I, which is quite a skewed distribution. The mid-year grades were very similar to the final grades with a Pearson Correlation Coefficient of .822. Only 18 of the 174 students improved their grade during the second semester (10%), and only 2 students (1%) changed from failing to passing during second semester. Within the two ethnic groups, success rates were similar.

Special Services to Target Group. Males were tutored by in-class tutors slightly more than females (19 males compared to 14 females). Few students in the target group used the after school tutoring available to them (mean of 1.5 sessions per student for the year), despite their low grades in Algebra . Only 15 of these students and parents came to either family math night and only a few participated in special counseling services. For example, one teacher reported inviting every parent in her Algebra class to math night, but not one showed up.

Additional Factors Affecting Progress. The teachers responded to a written survey at the end of the first 6 weeks to identify students doing poorly and to suggest what they believed were probable causes. The teachers reported that attendance, language difficulties and attitudes (interest, effort, consistency) were causes for the students' poor performance and that the curriculum was either too fast or "not connected" enough for about two thirds of the students. At the end of the first semester, 88 students had been absent more than 10 days and 31 students had been absent more than 20 days. Teachers reported using the classroom tutors to work with the frequently absent students to try to catch them up.

While most students seemed to respond to individual tutoring, comment cards by the tutors frequently describe "do nothing" attitudes and also cite poor prior knowledge as a problem for the students. A common response to a student survey near the end of the year is "My friends don't really think math is important for me but then, they don't really care. My family on the other hand, thinks math is very important for me." All of the D and F students were strongly urged to come to summer school free of charge, but barely one class was obtained.

Overall academic progress of these students is indicated by 102 out of 174 having at least 2 grades of D or F at the end of first semester. A serious concern for teachers during the second semester was how to organize instruction to benefit the minority of passing students and yet also engage the majority who were failing and understood little of the lesson.

Summary and Implications

The students in this study appear to have career ambitions which are inconsistent with their performance and effort in mathematics. While students express personal beliefs about the importance of mathematics, their daily actions seem to be influenced more by the popular peer attitude of doing little school work rather than personal or family values. These students do not exhibit the personal qualities of persistence, motivation to seek help, and consistency of effort necessary to succeed in college preparatory mathematics. Additional strategies and methods to promote better attitudes and increase attendance and effort of the part of the students are needed.

From all indicators, language difficulties is a limiting factor for many of these students and for teachers in communicating with students and families. Most of the teachers in the school are not from the target ethnic groups. Teachers remarked how tutors who spoke Spanish developed an immediate relationship with the Hispanic students. A simple phone call home to alert a parent of a problem must usually go through an interpreter, and many teachers admit to "just not bothering."

In summary, it appears that a team of extra support persons and services, a curriculum in line with the NCTM Standards, and a motivated and supported high school staff was not successful in helping many of the minority students succeed with Algebra I during the first year of this project. It is hoped that they will succeed in Algebra I and Geometry within three years. From this initial effort, it would seem that successful strategies for helping these students have either not been identified or that the reform program was not fully or well implemented by the partners. Many partners expressed that this was definitely a learning year about their role and responsibilities. Students and teachers expressed dissatisfaction with the curriculum, which is very language dependent. Revisions in all of the areas of emphasis are planned for year two.

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The Negotiation of Social Norms in a Mathematics Class

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The purpose of this research was to examine the negotiation of social norms in a university mathematics problem solving course. In this problem centered learning environment we find that college students with many years of traditional teacher centered mathematics classroom experiences negotiated a rich learning environment in which problem solving, sense making, and collaboration were important.

The focus of this study was a problem solving course for undergraduate mathematics education majors titled Problem Solving in Mathematics in which problem-centered learning was used as an instructional strategy (Wheatley, 1991).

> Problem centered learning has three components: Tasks, groups, and sharing. In preparing for class a teacher selects tasks which have a high probability of being problematic for students - tasks which may cause students to find a problem. Secondly, the students work on these tasks in small groups. During this time the teacher attempts to convey collaborative work as a goal. Finally, the class is convened as a whole for a time of sharing. Groups present their solutions to the class, not to the teacher, for discussion. The role of the teacher in these discussions is that of facilitator and every effort is made to be nonjudgemental and encouraging. (Wheatley, 1991, pp. 15-16)

The purpose of the research was to study the negotiation of social norms and how learning environments which promote meaning making are fostered (Trowell, 1994). The explanations constructed by the researchers to explain the beliefs and actions of this problem solving course were informed by an interactionist and constructivist perspective (Blumer, 1969; Bauersfeld, 1988, 1991; von Glasersfeld, 1989).

Problem Solving in Mathematics This class was conducted in a significantly different manner from a conventional mathematics course where the instructor lectures. In this course the students solved problems without any prescribed methods having been shown and then they presented their potential solutions to the class for discussion. The students became responsible for determining the viability of their solutions rather than depending on the teacher to tell them whether they were right or wrong. Many opportunities existed for peer interactions and the negotiation of meaning. The instructor did not dominate by lecturing, being highly prescriptive, or being the final authority when different answers were presented.

Negotiation of Social Norms Negotiation is a mode of communication in which individuals' attempt to make sense of each others' statements. Social norms are"taken-to-be-shared beliefs (and actions) that constitute a basis for communication and make possible the relatively smooth flow of classroom life (Cobb, Yackel, & Wood, 1993, p. 27)" - they defined the actions and interactions in this learning environment. The social norms are not seen as defining the mathematical meanings that each participant constructs, but social norms do influence the mathematics and problem solving that are possible in mathematics classrooms.

Voices and Research Procedures Three different groups were given voice in this research - the instructor, the students, and the researchers. Each of the class sessions were video recorded. In addition to recording each class, field notes were taken by a mathematics educator who was present at every class session. Immediately following each class session, the instructor shared his ideas, reflections, and thoughts concerning the class in video recorded sessions. Four students from the class were also chosen to be interviewed regarding their experiences in the problem solving class. Each student was interviewed individually five times throughout the semester with each session also video recorded. Listening to different voices was necessary in making sense of the beliefs and actions of the participants in this course.

Constituting the Learning Environment The learning environment in this classroom was interactively constituted; the actions and beliefs that came to be their problem solving orientation resulted from on-going negotiation. Social interaction is recognized as a activity in which participants form their meanings and not simply an arena for their activity (Blumer, 1969). Together the students and instructor formed their Problem Solving in Mathematics community with each individual participating in this construction.

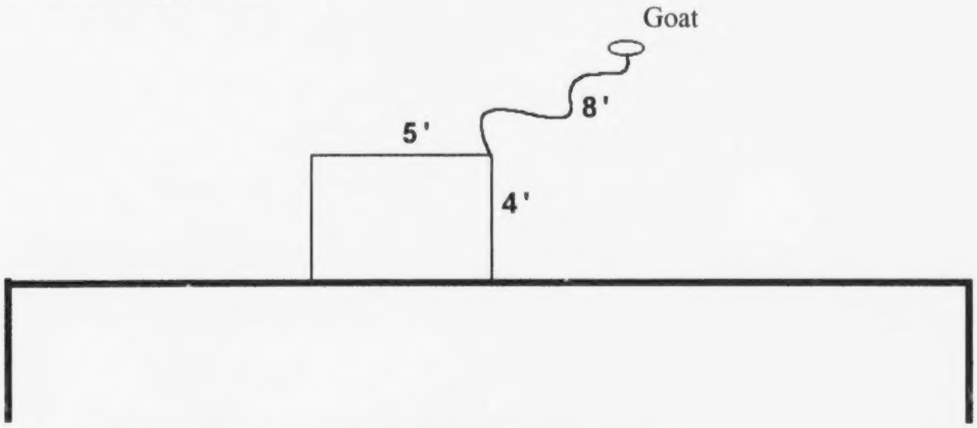
Although each mathematics class and their respective negotiated norms are unique, the norms in a conventional mathematics class are rather predictable. The instructor is generally recognized as the authority - the one who presents methods to be used and decides if a solution is correct. The students' success in such a class is often based on figuring out what the instructor wants and not necessarily upon making sense of mathematics. Each student in this problem solving class entered this course with a set of beliefs developed from at least fourteen years of such mathematics classroom experiences. The social norms of this problem solving class were significantly different from a conventional mathematics class. The negotiation and re-negotiation of norms began with the first class and continued throughout the semester. Not only did the students have opportunities to learn mathematics, they learned about doing mathematics.

The Social Norms of Problem Solving in Mathematics The norms that were negotiated in this class were established in the context of problem solving, not before hand. Problem solving is viewed as synonymous with doing mathematics and can not be separated or distinguished from learning mathematics. Mathematics tasks were selected which the instructor felt had the potential of being problematic. The goal of the course was for students to learn to solve nonroutine mathematics problems, drawing on algebra, geometry, number theory, calculus, and probability. Students had opportunities to solve problems/do mathematics in order to learn and give meaning to their experiences. Mathematics was viewed as a personal activity with opportunities for each person to construct their own mathematics (Wheatley, 1991).

The set of social norms negotiated in Problem Solving in Mathematics included collaboration, intellectual autonomy, students devising their own methods, students determining the viability of their solutions rather than the instructor, students initiating the presentation of multiple solutions, students focusing upon heuristics rather than answers, and an expectation for all solutions and ideas to make sense. The successful negotiation of these social norms was effective in establishing a rich mathematics learning environment.

In creating this learning environment, communication played a fundamental role in student learning. Communication between all members - between student and student as well as student and teacher - was critical. There were small group discussions and whole class discussions. These discussions provided opportunities for each student to present his/her ideas and solutions in an environment in which his/her thinking was valued, respected, and expected. These classroom practices were negotiated explicitly and implicitly. Ideas and solutions were discussed and defended in a collaborative environment.

On the first day of class the students were asked to work in small groups to solve the following problem.



What is the area over which the goat can graze?

With this action, the instructor immediately began negotiating that mathematics was a personal sense-making activity and that Problem Solving in Mathematics was to be problem-centered rather than teacher-centered. As the students began working together on this first task, they began negotiating the learning environment. They attempted to get the teacher to tell them the "correct" answer or procedure. When he did not respond to their questions as they expected, the participants began developing taken-to-be-shared social norms and meanings. The negotiation of social norms in this class was ongoing; the teacher and students continued to make sense of this class.

Assertions The instructor's stance and beliefs about mathematics and mathematics classes were significant in determining the social norms that evolved. His belief that mathematics is a personal activity which is experienced as students are engaged in problem solving, influenced the way in which he began negotiating the learning environment. From this study, we find that an instructor should listen carefully to students in an attempt to infer their mathematical constructions and beliefs. Students have valuable and insightful ideas, strategies, and heuristics and should have opportunities to discuss and make sense of these ideas in an environment of collaboration and challenge. As students have opportunities for this, an instructor also has opportunities to make sense of each student's mathematics and beliefs and can therefore better chose activities and potential problems for students.

We also find that a class has the potential to form an intellectual community. Students in Problem Solving in Mathematics began to act autonomously in the learning environment they negotiated. For example, a student would decide without consulting with the instructor to present their solution on the board. Students could share and extend their ideas freely and spontaneously. Students worked in an environment of collaboration rather than competition. Students questioned each other with an attitude of curiosity as they attempted to make sense of mathematical tasks being described by others. We find that given opportunities, students can be successful in solving problems when sense making is the goal. At the end of the course, most students were able to solve more difficult and complex mathematics problems such as those listed below:

> Soldiers Problem A column of soldiers 25 miles long marches 25 miles a day. One morning, just as the day's march began, a messenger started at the rear of the column with a message for the man at the front of the column. During the day he marched forward, delivered the message to the first man in the column and

returned to his position just as the day's march ended. How far did the messenger walk?

Spheres Problem A sphere with a radius one foot is rolled into the corner of a room so it touches the adjacent walls and the floor. A larger sphere is then rolled into the same corner and it touches the same walls and floor and also the original sphere. Find the radius of the second sphere.

Angle Bisector Problem Write the equation of the bisector of one of the angles formed by the lines $3x + 4y = -1$ and $5x + 12y = 2$.

Discussion The analysis of Problem Solving in Mathematics suggests that students at a university level can participate in negotiating social norms of what Richards (1991) calls *inquiry math*. The students in this mathematics class negotiated a rich mathematical learning environment even though each had participated in at least fourteen years of traditional mathematics classrooms. The students' beliefs are extremely important in this ongoing process as they are active participants in constructing the learning environment. The instructor plays an important role in this negotiation as s/he listens intently to students in order to make informed professional judgments. Negotiating social norms means providing opportunities for all participants to reflect upon their beliefs and actions and for them to have opportunities to form meanings that assist them in their interactions with others as they become engaged in making sense of their mathematics.

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EXPLORING THE SOCIAL IN SOCIAL CONSTRUCTION

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This report documents the mathematical and non-mathematical understandings constructed by a group of eighth grade mathematics students working on a series of problems dealing with the concepts of factor and multiple. The analysis suggests that both the mathematical beliefs and social knowledge students bring with them strongly affect the kinds of mathematical understanding which they construct. An analytic framework based on the hermeneutic tradition is used to emphasize aspects of group work which need specific attention in order to support the social construction of useful mathematical knowledge.

Introduction: Methodology and Theoretical Framework

It is now almost axiomatic in mathematics education to assert that students construct their own knowledge. Recent refinements to this basic constructivist tenet include a focus on the social nature of knowledge construction, and on teaching methods relying heavily on mathematical discourse among students, particularly in groups, as a means by which knowledge can be constructed and tested. Careful analysis of the processes by which mathematical understandings are socially constructed has only begun. Much work remains to be done in framing the problems within larger philosophical or theoretical viewpoints and in addressing the problems through empirical study.

This report summarizes our preliminary analysis of the social construction of the concepts *factor* and *multiple* by four eighth-grade mathematics students, focusing on the effects non-mathematical understandings had on this construction. The data were collected as part of a larger project which addressed the general issue of mathematical communication in middle school classrooms. The project team worked with this classroom for a full year, visiting several times per week. Some two- to three-week units were selected to be videotaped in their entirety. The data reported here are from the end of one such unit, and the particular group discussed was videotaped because one of the target students on whom the project had focused was a member.

In framing the analysis, we drew upon the hermeneutic tradition as represented by Heidegger and Gadamer. This tradition views understanding as fundamentally dependent on a *horizon*, or set of preconceptions which are always culturally embedded and often transparent. These preconceptions are the basis upon which we must approach anything we wish to understand; they make it possible for us *to know*. However, our horizons are expanded as we open ourselves to possibilities (e.g. other interpretations, other "readings" of the situation) and test these possibilities for personal meaningfulness. We cannot wholly step out of our preconceptions to obtain an unprejudiced view of truth; but neither must our prejudices necessarily remain unexamined and un-open to possibilities. Understanding comes within this hermeneutic circle, this interplay of foreknowledge (preconception) and possibility.

Applied to the construction of knowledge within a group, this analysis suggests that the understandings a group constructs will be constrained by individual members' foreknowledge and will grow through the group's considering the possibilities individual members (or the wider culture) provide. Understanding develops, then, not from accepting or adopting others' views uncritically, but from bringing others' views into meaningful relation with our own. This parallels what Polya has termed the "moral qualities" required to do mathematics, namely, "intellectual courage, intellectual honesty, and wise restraint" (Polya, 1954, p. 7-8). This means that group members must be willing to take seriously both the preconceptions and the possibilities brought to the group by all its members, and be willing to expand their own horizons. Our analysis of these data suggest that when the group members failed in this regard, their mathematical understanding fell short of the teacher's expectation.

Results and Analysis

Background

Prior to the group work session discussed here, the class had worked for several days on the concepts of factor and multiple in the usual context of natural numbers. The teacher, Mr. Scott, then wanted to provide an opportunity for students to work together, discuss ideas surrounding factor and multiple, and extend the ideas to algebraic expressions as a way of deepening and extending their understanding. The students in the class were placed in teacher-chosen groups and were given a series of problems on a handout, along with a mandate to make sure everyone in the group understood the answers. Mr. Scott specifically instructed students not to work individually and compare answers at the end, but to discuss the problems as they worked. He encouraged groups to rely on their own resources, and agreed to answer only three questions per group. Mr. Scott made periodic visits to the groups to award points for working together. He also chose a problem from the worksheet for each group to present to the class.

The Problem Solving Sessions

We separate the time the group worked together into two phases. Most of the progress made toward both individual and group understanding of factor and multiple occurred during the first phase, which lasted about one class session. During the second phase, the non-mathematical understandings which were (re)constructed by the group significantly and negatively affected the group's mathematical progress.

Phase 1. The group began with somewhat informal but still useful descriptions of factor and multiple, such as that factors are what "make up" a number, and multiples are "everything times" the number. Initial explanations relied heavily on specific examples, and were strengthened early in the session by a group decision that factors were generally smaller than the number, whereas multiples were larger. This represented a move away from a structural meaning (however informal), but still represented a type of knowledge which served them well when answering questions about whole numbers. However, they soon met problems which required that they discuss factors and multiples of algebraic expressions such as 7x, and their foreknowledge based on natural numbers proved insufficient. In the absence of a more structural definition, they began to reverse the meanings of factor and multiple; this was a first critical point in the joint construction of knowledge. They resolved their confounding of factor and multiple by referring to an example in their textbook. They also turned to a bulletin board display which gave them some help in understanding the algebraic expressions (eg. $x + x = 2x$). Through this, they were able to begin the work of re-constructing meaning for factor and multiple in the context of simple algebraic expressions. The group provided some insightful and creative analysis during this period. For example, they had a sustained discussion of whether $\{2m, 3m, 4m\}$ or $\{m^2, m^3, m^4\}$ would be the better list of multiples of m, eventually deciding that either list was correct. However, they had some difficulty extending this result to obtain multiples of $2w$ and ended their first session unsure how to proceed on the next problem of the set. Still, at the end of this session, all group members remained more or less involved, and their work reflected an adequate understanding.

Up to the end of this first period, the group had been functioning in ways which supported understanding. That their understandings were unstable is sensible considering that this was their first attempt to extend the ideas of factor and multiple to algebraic expressions. The group was free to create mathematical understanding because they had, with the teacher's help, expanded their horizon about what it meant to do mathematics. Although a group leader quickly emerged, all group members began the session open to the possibilities afforded by one another's views. When disagreements occurred, the group spent considerable time resolving these conflicts. In short, there was promise in this group of the legitimate construction of meaningful mathematical ideas in which each member had legitimate ownership.

Phase 2. The second part of the problem solving session, during which the group considered 18 of the 20 problems, can be characterized as a return to more traditional classroom behaviors. A few examples will suffice to contrast it with Phase 1. First, whereas during Phase 1 the group had rejected any help from the teacher, near the beginning of the second period of group work, they were ready to ask the teacher a question; they had decided they didn't know "what they were doing." Second, the understandings developed during this period were more seriously flawed; by the end of the second period, the group was agreeing, for example, that if x were divisible by 5, it could not be divisible by 3. Third, the group made more and more use of the understandings they had developed for natural numbers, and the group leader interpreted all problems in ways which could be solved by those informal methods. They became more focused on arriving at answers quickly; discussion had stopped. For example, when asked to write an algebraic expression which contained a variable and was divisible by 3 (e.g., 9x), Nancy suggested "m is divisible by 18" as a solution, and the group accepted it without comment. Finally, as they discussed the problem they were to present to the class, they fell back on examples using natural numbers to justify their statements. For example, given that $3y = k$, they would argue that y is a factor of k by substituting 5 for y and 15 for k, and verifying that the resulting statement was true. In summary, Phase 2 was characterized by a return to routine doing of problems, a focus on obtaining solutions, and little if any effort to make meaning of what the group was doing.

Analysis

We suggest that the changes in the way the group functioned were largely a result of socially constructed understandings, or more properly, *re*constructed understandings. As their work together progressed, group members began to reconstruct the meaning of the task in which they were engaged, and the roles they played in that task. This construction was in fact a recapitulation to the traditions of the usual mathematics classroom. With this mutually accepted foreknowledge in place, the horizon for mathematical understanding was extremely limited. We discuss this recapitulation in the space remaining.

The construction of a traditional classroom microcosm within the group affected at least three aspects of group functioning. Nancy assumed the role of teacher as well as leader; the group re-interpreted their task as "getting done;" and Edward became marginalized as an incompetent student. Nancy, the group leader, was generally seen as both competent and responsible by teacher and students alike. She quickly emerged as the leader and as the one whose goals would be ultimately accepted by the group. Over time, the preconceptions of the individuals in the group, based on their membership in the larger culture of the classroom, (re)constructed Nancy as an expert, one who knew the answers. This role was supported by the teacher. In his first visit to the group, he commented, "Good job, Nancy, for helping Edward."

Nancy's role as leader was supported by the group's eventually adopting her primary goal: getting the problems finished. Our work with her through the year suggested that she usually saw this as her major task in mathematics class. Mr. Scott also inadvertently supported this goal by telling the class approximately how many problems they should get through in a period. During Phase 1, the group addressed 2 of 20 problems, fewer than the 7 problems Mr. Scott recommended. Late in the second period, Nancy commented, "Guys, we can do problem 14, we can be good for the day." When the teacher asked how the group was doing, Nancy replied, "Better, we're on number 13 now."

Because getting problems done in class emerged as an important goal, and because Nancy could only trust herself to get the job done, she assumed the role of "teacher" in the group. Nancy's leadership kept the discourse focused on getting answers, though she herself felt she understood and could explain concepts to the others ("Does everyone understand that? You should be finished with it because I just said it [the answer]."). Nancy's explanations were often in the form of questions, designed to prompt the correct response, e.g., "So, Edward, what are the multiples of 5? OK, so 25 is a multiple of 5. Write that down." Nancy used the sort of "cued elicitation" noticed by Edwards & Mercer (1987); she asked questions which lead quickly and directly to answers which could be written down.

During Phase 2, then, Nancy and the other group members came to adopt a traditional classroom as their model for group work--a classroom in which the instructor had primary responsibility for mathematical knowledge, felt compelled to cover material, and tested understanding primarily by probing for correct answers. With this came a shift in responsibility. During Phase 1, group members shared responsibility for their work by attempting to make substantive contributions to solutions. By contrast, through most of Phase 2, Nancy had almost sole responsibility for solutions, and other group members only took turns reading the questions. Similarly, *understanding* came to mean agreement with Nancy's solution. Nancy frequently asked if the other group members understood, and followed up with another explanation of how she arrived at the answer when she sensed they didn't. She complained to the teacher, "I understand this, but they won't listen to me."

In light of this recreation of a traditional classroom, it is disturbing but not surprising to note that Edward became marginalized as incompetent in mathematics. This is particularly troubling because he clearly demonstrated understanding during Phase 1. On the two separate occasions in Phase 1 when Edward and Nancy disagreed, Edward was right about the mathematics they were doing. By contrast, Nancy demonstrated lack of understanding at several points throughout both phases. When the ground rules for group work as originally established by the teacher supported Edward's views being taken seriously, Edward participated in ways which demonstrated understanding often superior to Nancy's. However, he did not receive recognition from the group that he was correct. Whether because his contributions were not valued, or because the more traditional goal structure had already marginalized him, throughout Phase 2 he grew gradually more distant and eventually adopted his place as the student who could not do mathematics. At one point he said to the teacher, "They're telling me how to do this," as though he believed this would communicate proper group functioning. Edward's role was also supported by the group; most of

Nancy's utterances directed toward Edward provided an answer and ended with "Write that down." By the third day, a good deal of time was being spent by Nancy trying to explain problems to Edward, with Edward paying only nominal attention to what was going on. Nancy began to discipline Edward as a teacher would: "Edward, what did I just say? Edward, please work, or we'll never get through." By the time the group was working on their presentation, Edward was bluntly stating, "I hate this." This stood in marked contrast to his engagement during Phase 1.

It is interesting that the goal structure adopted by the group and the concomitant relegation of Nancy and Edward to their expected roles ultimately become problematic for the group. Because the group had to explain one problem to the class, and because each group member was supposed to understand the problem, the group spent a good deal of time on the final day worrying about how Edward would explain his part of the problem. Having adopted a goal structure which systematically locked him out, the group had no way of opening the door to his real understanding when it was needed. In fact, during the presentation, Edward was unable to explain how the group had arrived at an answer, and in fact simply read the list of statements the group had produced.

Conclusion

The focus of this report has been the social influences which affect construction of knowledge during group work. The group we observed constructed overall a fairly impoverished understanding of factor and multiple as they related to algebraic expressions. It is not uncommon for group work to break down in this way, and other analyses can shed light on the processes involved. Our purpose here is to highlight the complex tapestry of social processes which underlie the construction of common knowledge, and begin a more careful study of how they affect knowledge construction. It seems clear that these processes need to be addressed as teachers attempt to support the social construction of knowledge.

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Students' Attitudes and Beliefs

RESEARCH PAPERS

Empowering Students To Talk About Mathematics In A Seventh-Grade Classroom 115
Verna M. Adams

Confidence In Answer Keys Among College Calculus Students: Finding Out The Truth About Santa Claus 122
Hylie T. Barton

Enhancing Student Interest In Mathematics Using Integrative Curricula 129
Mathew Mitchell

The Effects Of Psychosocial Variables On Middle School Student Problem-Solving Achievement In Mathematics 136
Janice L. Nath, Susan E. Williams, and Hersholt C. Waxman

Justifying The Reasonableness of Answers: Processes Of Middle School Mathematics Students 143
Susan E. Williams and Juanita V. Copley

ORAL REPORTS

Voicing The Voiceless: A Mathematics Teacher's And A Student's Views And Beliefs About Teaching And Learning Mathematics 150
Evangelina Diaz-Obando and Elizabeth M. Jakubowski

Student Authority In The Classroom 151
Laura Coffin Koch

POSTERS

When Do The Equilateral Triangles And The Congruency Get To Be Boring? Introduce To Deltahedral Models 152
Eugenio Diaz Barriga Arceo

EMPOWERING STUDENTS TO TALK ABOUT MATHEMATICS IN A SEVENTH-GRADE CLASSROOM

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This study examined the development of student initiated exploratory discourse in whole-class discussions in a seventh-grade mathematics class over a two year period. During the second year, students began to offer and refute their own theories, and questioned, extended, or modified each others' ideas during whole-class discussions. Systematic analysis of whole-class discourse patterns suggests that elaborative discourse, initiated by the teacher, set the stage for the development of exploratory discourse. In elaborative discourse, the teacher's control over knowledge is partially shared with the students through the teacher's efforts to build the discussion on student ideas. Exploratory discourse occurred when the teacher was willing to set aside her own agenda. The benefits of whole-class exploratory talk are questioned when the discussions are not brought to closure.

Efforts to create classrooms consistent with recommendations in reform documents (NCTM, 1989, 1991) require teachers to face the prospect of changing the discourse patterns in their classrooms. Although examples of how that discourse might be structured are available (see, for example, the video Changing Practice: Teaching Mathematics for Understanding provided by the National Center For Research on Teacher Learning), information on classrooms in transition are needed in order to provide markers for interpreting progress toward those goals.

The current attention on communication in the mathematics classroom is supported by studies of communication patterns in schools (Barnes, 1992; Barnes, Britton, Torbe, 1990; Edwards & Mercer, 1987). These studies suggest that control of knowledge is related to communication patterns and learning. Barnes (1992) states: "The more a learner controls his own language strategies, and the more he is enabled to think aloud, the more he can take the responsibility for formulating explanatory hypotheses and evaluating them" (p. 29). He argues that students are capable of controlling knowledge if the social context supports it. Barnes (1990, 1992) labels talk in which students are in control of knowledge *exploratory*; that is, their talk explores a range of possible explanations and students develop one another's ideas, bringing together their knowledge with current experiences. In many classrooms exploratory talk seldom occurs during whole-class discussions, but often occurs in small-group interactions. When a teacher is present, students' talk shifts to a focus on right answers, which Barnes labels *presentational* talk. Generally, in episodes of presentational talk students do not initiate discussion, focusing instead on meeting the teacher's criteria of correctness. Barnes (1992) suggests that "as the form of communication changes, so will the form of what is learnt" (p. 15). From the communication pattern in the classroom, students not only learn whether they are expected to memorize details or reason about evidence, but also their role in formulating ideas.

This paper discusses the development of student exploratory talk in a seventh-grade mathematics class during whole-class discussions and attempts to explain how the social context of the classroom contributed to increased student-initiated discourse. The research discussed in the paper was part of a larger project that focused on problem solving and communicative interaction in middle-school mathematics classrooms.

Methodology

The seventh-grade mathematics teacher described in this study was observed over a two year period. Data consisted of classroom observations recorded on video tapes and in field notes. Events in the classroom were observed at least twice a week, with daily video taping of 2 to 3 instructional units during each school year. Each of these units were 2 to 4 weeks in length. Audio-taped interviews with students explored their beliefs about mathematics, their ways of knowing and their attitudes about participation in the class. Audio-taped interviews with the teacher explored the teacher's perception of the class and perception of the reasons for increased student involvement in discourse.

Analysis of the video-taped whole-class discussions are the focus of this paper. Analysis procedures drew on a sociolinguistics approach to discourse analysis (Stubbs, 1983). Stubbs suggests that the study of overt signals of discourse organization is a way of studying topics of sociological interest. In order to explain the ways in which this teacher changed the social context of the classroom to support exploratory talk by students in whole class discussion, comparisons were made between first year data and second year data and comparisons were made between segments of data within the second year. Initiation and response patterns in whole-class dialogue were used to identify types of discourse patterns. The underlying structure in each of these types of discourse was examined for ways in which the teacher signaled an increasing value placed on exploratory talk.

Analysis

On the surface, the change in communication patterns in this classroom might be attributed to a change in the way the teacher managed classroom discussions: the teacher moved to the back of the room in order to allow students to come to the front of the room to discuss their solutions to problems at the overhead. That procedure by itself, however, does not explain why the comparisons of second year data to the first year showed an increase in the amount of exploratory student talk in which students offered and refuted their own theories, and questioned, extended, or modified each others' ideas during whole class discussions. A more complete explanation lies within the discourse itself. The focus of the following discussion is the components of the classroom discourse that appear to have contributed to the increase in exploratory talk.

Episodes of student initiated discourse were often embedded in discourse that I call *elaborative*. In contrast to exploratory talk, elaborative talk is teacher initiated, but is more complex than presentational discourse. In the following example of elaborative discourse, taken from the beginning of a unit on number theory, the teacher keeps the dialogue going by accepting students' responses and, at the same time, making the group responsible for modifying and expanding the definition of factor. (All names of students are pseudonyms.)

1. Teacher: Alright. Who knows what a factor is? Who can tell me what a factor is in your own words? Think about it. (pause) Maury, are you thinking? (pause) Okay, lets hear Linden's definition of a factor.
2. Linden: The answer, um, that you get from a subtraction or mult--er, no its not. It's a , a--It's a division problem? I forget. I need to think.
3. Teacher: So, it has something to do with one of the operations.
4. Linden: You know like common factor or something like that.
5. Teacher: Uh huh. Okay, lets hear Terry's definition and maybe it will come back to you. You're in the neighborhood.
6. Terry: He is? That's different than mine. I think that factors are a whole bunch of numbers.
7. Teacher: A whole bunch of numbers?
8. Terry: Well they're like, different numbers for the factors.
9. Teacher: Okay, that's a good beginning. Take it from there, Brian.

At Line 1, the teacher asks students to use their own words, emphasizing that she expects them to "think." At Line 3, she ignores the student's insecurity and validates his response. The student then attempts to indicate that he remembers something about factors (Line 4). The teacher is accepting and encouraging (Line 5) and moves on to another student. Terry (Line 6) interprets the teacher's response as an assessment and shows surprise. The teacher's questioning repetition of Terry's words (Line 7) provides an invitation to expand his explanation. At Line 9, the teacher accepts his response and calls on another student in a way that suggests that the activity of describing a factor is a group responsibility. This discussion continued for a total of seven minutes with the teacher orchestrating the discussion, asking for examples, asking students if they wanted to change their definitions and how they would change them after hearing other students' responses.

Elaborative discourse occurred almost every day during the second year. Within those episodes of elaborative discourse, the teacher set the stage for student initiated exploratory discourse to occur during whole-class discussions. She encouraged respect for each other's efforts ("Hold on just a minute, she's doing it." "Just let him finish though. Just one second, okay--because it's hard to think and answer questions at the same time." "Let's listen to David and then we'll come back." "Everybody makes mistakes."), acknowledged individual contributions ("There you go--an example using Barbara's idea."), and encouraged students to interact ("Now, would anybody like to disagree with that?" "Okay, any questions for Terry about that one?"). The teacher supported individuals who were hesitant to participate at the overhead ("We'll help you out."), and encouraged everyone to talk about their thinking ("Tell us what you're thinking as you do it." "Okay, can you say it in your own words?" "Good thinking people.")

The teacher included herself as a member of the group ("Could you do it for us?"), but still acknowledged her role as authority figure ("Boy, I'm such a slave driver."). She allowed choice ("Whichever works for you." "And I'm going to let you choose your own number."), and she indicated that she valued talking about their ideas more than being right:

> Teacher: Go ahead and tell us what you're thinking because I think it's good. I want to hear it.
>
> Karl: But it's wrong.

Teacher: Well, lots of our ideas are wrong. Just tell us what you were thinking and then tell us why it's wrong.

Episodes of exploratory talk began to occur within episodes of elaborative discourse when the teacher relaxed her own agenda and encouraged elaboration of student ideas. Students began to indicate that they had ideas that they wanted to share ("Well, I have an idea that I just thought of." "OK, I don't know if this is Kevin's or not, but, because I kind of didn't understand his--") The changes in the communication patterns led to an event which took a full class period and part of a second period in which seven students (two girls and five boys) volunteered to share their theories about number patterns with the class. Some of the students attempted to make modifications on the first student's theory and other students offered new ideas. As students talked, the teacher continued to support exploratory discourse ("Well, what can we do with Steve's theory? It doesn't work every time, unfortunately, but it was a great idea. Could we modify it in some way? Could we put some limitations on it so it would work?"). This event was the most notable shift to exploratory talk during the second year and raises issues that need further investigation. If, for example, the teacher does not recognize productive directions for the discussion, which she can help bring to closure, and the students themselves cannot bring the discussion to closure, what is learned? Do students gain a better understanding of the process of doing mathematics?

Discussion

In this study, the teacher was purposefully working at increasing the amount of student talk about mathematics in her classroom. Also, the point of focus for that discourse was on explanation and justification. This study verifies Barnes' claim (1992) that teachers can change the patterns of discourse in the classroom and identifies ways in which a teacher encouraged the development of exploratory talk in whole-class discussions. During the second year of the study, the teacher's communication patterns reflected greater support for student talk and exploration of ideas and the students responded in a positive way. Although data during the first year showed the teacher responding enthusiastically to student ideas, students seldom responded to each other's ideas.

Three types of discourse were identified on the basis of an analysis of initiation and response patterns in whole-class discussion. Two of these types (presentational and elaborative) were teacher initiated; the third type (exploratory) was student initiated. Exploratory discourse was encouraged by the teacher during the second year. The teacher, instead of assessing the ideas presented by the students, managed the discourse, making positive assessments of the thinking and of the discourse processes being used by the students. Even when the discourse could be labeled *presentational*, usually during episodes in which the class was practicing a procedure, the teacher often asked students to generate their own problems. Thus, even during the practice of procedures , the social context of the classroom suggested to students that they could make decisions and have some control over their own learning.

Elaborative discourse was identified as a pattern of teacher initiated discourse in which the teacher encouraged individual effort within the context of group construction of concepts. In elaborative discourse, the teacher partially shares control over knowledge with the students through the teacher's efforts to build the discussion on students' ideas. The distinction between elaborative and exploratory discourse can be made on two levels; in exploratory talk, the students initiated the discussion and the teacher did not always know where the discussion was headed. The second factor is critical in the teacher's evaluation of the benefits of exploratory talk. Although students were able to formulate hypotheses, they needed help in evaluating them. Because the teacher did not always recognize where the discussion was headed in terms of the potential mathematics that could be learned, the teacher could not always help students bring the discussion to sufficient closure for students, or the teacher, to have the feeling that something was learned from the discussion. (In informal observations of this classroom in a third year, exploratory talk, for the most part, was absent. For a discussion of the dynamics of change in this classroom, see Adams & Williams, 1994.) The study of this one classroom does not establish a pattern; however, it does raise the question as to whether or not elaborative discourse is a transitional pattern between presentational discourse and exploratory discourse in the development of exploratory talk in whole-class discussions. It also points to the need for further research on what is learned and how teachers successfully manage whole-class exploratory talk so that it is mathematically productive.

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CONFIDENCE IN ANSWER KEYS AMONG COLLEGE CALCULUS STUDENTS: FINDING OUT THE TRUTH ABOUT SANTA CLAUS

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Although one goal of mathematics education reform is to have students take ownership of their answers and the entire learning process, many students have learned not to trust their own intuition or processes over outside "authority." In this study, first-semester calculus students were video taped while collaboratively solving problems. Half of these groups were simply given a set of problems and were taped while solving them. The other groups were given answer keys which contained incorrect answers. With rare exceptions, students with these keys would alter their correct answers to match those on the key. The observations from this study point to some interesting questions about the issue of authority. When confronted with an answer key, students almost inevitably reverted to the mentality that what they thought was not as important nor as correct as whatever the authority said.

Many students see mathematics as little more than a collection of algorithms and formulas. After applying these algorithms to problems, students come up with answers which are then checked by the teacher or against some form of answer key. Consequently, students often are not compelled to think carefully about or reflect on the procedures they used and the answers they obtained. In fact, there may be little if any effort made to give those procedures and answers meaning. Students may not be asked to explain their processes and reasoning or defend their result if it differs from that of a peer or especially that proposed by the teacher. In this way, students often learn and come to believe that success is measured by how closely their work aligns to that of the authoritative teacher or key. Unfortunately, by the time students graduate from high school, most have learned not to trust their own intuition or processes over the pronouncements of outside "authority." They have learned that success in mathematics is based on getting the correct answer in the correct manner, not on meaning or understanding. (see Lampert, 1990)

One goal of mathematics education reform is to have students gain an understanding of the mathematics they are studying, to have them make meaning. While we are not saying there is no need to ask students to get correct answers, we do contend that these correct answers and algorithmic processes, when not accompanied by a sense of meaning, do not constitute learning mathematics. To be able to make meaning, students must break the mindset that the traditional authorities--teachers, textbooks, and answer keys--have some direct link to "Truth" that is inaccessible to the learners.

Method

The students in this study were enrolled in the fourteen week first-semester calculus course at a large private university. The class of approximately 240 met three times weekly with a professor in a large lecture hall. As a teaching assistant, I met an additional two days a week with approximately

100 of these students, divided into three equal sections. Within these small sections, each student was assigned to small collaborative groups with five other students. Students were encouraged to study regularly with their groups. As an incentive to motivate active participation in group work, students were told that each individual's grade would be raised if the group average increased significantly.

During the course, students did not have access to a solutions manual nor were answers provided in the back of their text (Hughes-Hallett, et. al, 1992). Instead, they were advised to use their groups as "interactive answer keys" on all homework assignments. We also assigned some group projects and quizzes to more strongly emphasize the need to work with a group.

Ten weeks into the course I began video taping problem-solving sessions. Groups were taped outside of class time during the tenth through twelfth weeks of the semester. This was not a required part of the course but was offered as an extra-credit opportunity on a volunteer basis. This credit was given based on participation and a good faith effort, not on getting the correct solution. I told the students that the purpose of the taping sessions was to observe their problem-solving skills employed in a group setting. Fourteen groups took advantage of this opportunity with an average of four members attending. A total of 65 students were taped solving one of two problem sets. Half of the groups were given answer keys which they were told to use in whatever manner they saw fit. Select answers on the keys were intentionally altered to be incorrect.

Immediately after the conclusion of each taping session, I asked the participating students to individually write a paragraph describing their taping session from their perspective. After all groups had been taped (about a three week process), the students were debriefed and told that the keys had been incorrect. I then talked informally with the students about their reactions and encouraged them to write a description of their feelings or responses as they reflected on their experience with an incorrect key. In addition to these responses written near the end of the course, as part of the regular course requirements, students were to submit weekly journal entries.

Results and Analysis

Even after several months without the influence of an answer key, with only one significant exception, those groups with a key devised a method to reach the answer given on the key, even if neither their answer nor their methods made sense. By contrast, those groups without the key asked more insightful questions about what their answers meant as well as the validity of those answers.

This paper will focus on three groups working on part b of the "candle problem"--one group without a key, one group that agreed with the key, and one group that disagreed with the key. As seen below, students were presented with a table of data concerning temperature and illumination as a function of distance from the candle. Because there was no algebraic formula from which these data points were derived, students had to use their understanding of average rate of change and concavity to solve the problem. Though the correct solution to this problem is -2.17 • F/ft, the answer key was intentionally altered to read -0.46 • F/ft, or the multiplicative inverse of -2.17.

Candle Problem

Alone in your dim, unheated room you light one candle. Dismayed, you walk directly away from the candle. The temperature (in degrees Fahrenheit) and illumination (in % of a candle power) decrease as your distance (in feet) from the candle increases. In fact, you have tables showing this information.

Dist (feet)	Temp (° F)	Dist (feet)	Illum (% cp)
0	55	0	100
1	54.5	1	85
2	53.5	2	75
3	52	3	67
4	50	4	60
5	47	5	56
6	43.5	6	53

b) What is the average rate at which the temperature is changing when the illumination drops from 75% to 56%?

Source: B. Osgood, et. al, *Instructor's Manual, Calculus: Preliminary Edition*

Group 1 Candle Problem, No Key

The first group discussed here works without an answer key. Although they work quickly, they seek to make meaning of their results and exemplify the reflection process we wish to see in our students. Their discussion of this part of the problem goes as follows (the italicized comments are those selected as revealing the reflective, meaning-making process):

E: O.K. What is the average rate at which the temperature is changing?
A: What is the average rate? So you want the derivative from 75 to 65....
E: Yeah, so that is • y over • x.
A: So you put it into that little formula thing, right? You want the average rate.
E: You want the average rate. So it'd be 75-56/3.
A: Yeah. The change in y over the change in x.
E: *Wait, wait, wait. It'd be negative 3 because the rate is negative, right? Wait, the rate could be negative. The rate of change would be negative because you go 2 minus 5. So it'd be negative, right? You gotta go the same way both times.*
A: Yeah, that's right. That's right.
E: O.K. Just checking.
A: 75 minus 56 over 2 minus 5. So -6.33.
E: Yeah. So it's saying at that time, it's changing at a rate of-- Wait a second, guys.
A: It's changing that much % per foot.
E: *But then it says what's the average rate at which the temperature is changing.*
A: Yeah. That's illumination. Now we gotta do temperature.
E: Wait a second. Wait. It's asking temperature. So we're supposed to do the temperature one.
A: Darn.
E: So it's 53 minus 47. So that was just another way of saying from 2 to 5 feet.
A: That was dumb.
E: They tried to trick us. O.K.
A: Dirty, dirty. Not good.
J: -2.16.

E: Yeah, -2.16. So at that time, it's changing, the temperature is going down---
A: Wait, what did you get?
E: -2.16667
A: *So what did that mean?*
E: *At that time, 2 to 5 feet, temperature is going down 2.1667 every foot. Average rate type thing.*

Although comments like "what did you get" demonstrate that these students value answers, questions such as "so what did that mean" show that they are not content to accept their answers until they have determined what those answers mean. Of course, we cannot be certain that, had they been presented with an answer key, these students would not have reverted back to valuing the "correct answer" (as validated by the key) above making meaning of their work. But, it seems clear that while working without a key, these students are not willing to accept an answer until they have made that answer meaningful.

Group 2 Candle Problem, Key, Group Agrees with Key

Here we examine the stereotypical group that is working with an answer key. As compared to both the previous group that did not have a key as well as the third group that disagrees with the key, there is a significant difference both in the length of their discussion and the level at which this group is operating. Clearly, this group is driven by the need to get an answer and making meaning of those answers becomes a secondary consideration. They ask no questions about meaning or reasonableness throughout their taping session (this is true for all parts of the problem) and instead simply turn to the answer key upon arriving at an answer for each part of the problem. Their discovery that their answers agree with those given on the key marks the end of their discussion. Because their goal is to get the correct answer rather than to make meaning of their work, the confirmation of the answer key signifies that they have clearly succeeded. In the following discussion, we look at their work on the part of the problem for which the key is incorrect:

S: If you took the change in illumination over the change in distance won't that give you the average? So 56-75 over---
J: It would be 6.3
M: Would it be negative?
S: It would be negative. I guess you're right there.
M: You get the same answer if you take all of them [the rates of change] and add them up.
S: Oh, did you do it that way?
M: Yeah....
J: [Checking key] The graphs are right. It's just part b.
S: [Rereads question, emphasizing "temperature"]
M: Oh, you guys.
J: You need to go to temperature.
M: So it's 2 to 5.
S: You go 47-53.5 over 2-5. Equals -2.166.
J: It's distance over temperature, not temperature over distance *to get the answer.*
S: We do distance over temperature?
J: Right. -3 over---

S: *But slope is rise over run.*
Mi: *But we're doing temperature against distance.*
S: *[after a long pause] O.K.*
J: So it's going to be -0.466.

Though S initially seems disturbed with J's methods "to get the answer," he does not ever demand that J explain the meaning of his processes. He apparently lacks the belief in his own work to press his objection. Consequently, although he never appears convinced, S ultimately concludes that they key must be correct and that his initial correct intuition is not reliable.

Group 3 Candle Problem, Key, Group Disagrees with Key

Here we observe a group that, after much discussion, correctly concludes that the answer key is incorrect. They do so, however, only after repeated attempts to get the answer given on the key and after much questioning about the meaning, both of their work and their solution. In their discourse community, they discuss the problem at hand and gain confidence that their methods are reasonable and therefore correct. Though we must keep in mind that this group was the exception, they do illustrate that an answer key need not be detrimental so long as the students will value meaning at least on a par with answers. Again, the italicized comments are particularly interesting in light of the students' attempts to make meaning and to reconcile that meaning with their cultural bias that tells them to trust in the authority of an answer key.

K: What did you start with? I got -2.2. [E takes key]
Kr: You go 54.5-43.5 and divide by 5. What is it? [Looks at key] *What did we do wrong?*
K: I don't know.
Kr: *How did they get -0.46?*
E: Maybe this is a trick. Check the answer key again. Let me see it.
Kr: *Maybe we have to do the temperature over the illumination.*
E: Well, they started at 2. So we didn't have to do these [rates of change from 0 to 1 and 1 to 2]
Kr: *There's no way it can be -0.46. If we have 1.5, 2, and 3---*
D: That's true....
D: *Something is wrong here.*
E: *I don't see how they could get that number.*
[D divides answer by 5, getting -.4 and shows answer to Kr]
Kr: *Oh, dude! But wait, you already divided by 5. Oh, I mean, but we did it right.*
E: *[looking at D's calculations] Oh, yeah, duh. That's where we got it wrong. We didn't divide it by 5.*
D: *Yeah, you take our answer and divide it by 5 and get the answer. But I don't know why you do that.*
E: *But we don't want to do that. Well, yeah, sure you do.*
K: Yeah, yeah.
E: *Yeah, because you get the right answer that way.*
D: *I still don't know why we do that....*
E: *It's because you get the right answer that way.*
K: *I don't understand.*
D: *Oh, [there's] 5 intervals.*
Kr: *But we already divided it by 5.*
D: *That's the average.*
B: *We already did that. To get the average, you take the answers and add it up and then divide. You get your average, right? So why would you divide again?*
K: *To get the right answer.*

Kr: *Yeah, because that gives us 0.46.*
K: *Because that's the answer....*
D: *Oh, I got it. We go 2-5 over 53.5-47. We did it backwards.*
E: *We do?*
Kr: Oh.
E: *How'd you get that?*
D: *I don't know. New insight.*
E: *Wait, watch... [points to slope formula of • y/• x]*
B: *Yeah, that's what I was just going to say.*
Kr: *You have to put y over x.*
D: *If you take the inverse of 2.2, you get 0.46.*
E: *The inverse?*
D: *Yeah, one over. Our answer is -2.2. If you take the inverse, you get -0.46.*
E: Wow.
D: *So that's how I determined we did it backwards.*
B: *But that would give us run over rise.*
Kr: *That should be right though, right. I think he's right. Wait, there's no way he's right.*
E: *I don't trust him.*
B: It's gotta be rise over run for your slope....
K: Did anyone try looking at just from 53.5 to 47?
Kr: *But if we do that you have -1.5, -2, -3. That'll be way above -1, so there's no way it could be...*
D: *I think they did run over rise instead of rise over run.*
B: But why? Why?
Kr: *I don't know. We're saying that we should be right on the problem.*
K: Well, I don't think they'd give us a problem and a faulty answer key though.
D: I don't understand it.
E: They did it backwards I think. Oh wait! Never mind....
D: We can't do number b. There's no way.
K: *I think on part b they messed up.*
D: *It's -2.2. It's gotta be -2.2. Nothing else.*
E: *I'm just gonna put the inverse of -2.2.*

This group is clearly torn between their belief that mathematics means getting the right answer--as confirmed by the authority of the answer key--and their notion of what constitutes reasonable work and a meaningful answer to this problem. We note a repeated cycle which begins with D proposing a method which generates the answer given on the key. Though the group is initially excited by this "new insight," their enthusiasm weakens as they begin to ask questions about the meaning of what D has done. Yet, the group members repeatedly assert that D must be correct because his methods "get the right answer." Perhaps their belief that an answer key must be correct is so strong that they want D to be correct so they can proceed peacefully with the remainder of the problem. But, ultimately, it is their need to make meaning of their answer that conquers their belief that an answer key is always correct.

After several minutes of further discussion, the students decide to leave their answer as -2.167, despite what the answer key said. As the students concluded, I talked with the students about their answer, and our conversation proceeded as follows:

Me: Do you guys think you're right?
ALL: Yeah.

Me: Do you believe it?
D: Yeah, except for b.
Me: You don't believe your b answer?
E: We don't believe the answer key's answer. We believe our b answer.
Me: Why is yours right?
D: We did it right.
K: We discussed it.
Kr: We talked about it.
E: We did it the way we were taught.
Kr: We did it the way you taught us.
B: And *we went through every reason it could be wrong.*

These final comments are very revealing about the influence of the collaborative community. Their discussion of the problem led them to feel certain their answer is correct. In other words, they constructed mathematical reality independent of, and in fact even contrary to, that which the authority told them is true.

Conclusion

Clearly this last group shows that answer keys are not inherently bad. The value of an answer key is dependent on how it is used. However, if students use an answer key to replace the need to make meaning and revert to the mentality that "the correct answer" and not understanding is the measure of mathematical success, then certainly the presence of an answer key serves only to stunt the student's growth.

In conclusion, I turn to one student's journal entry written after learning that the key had been incorrect. He writes:

> I still don't quite know what to make of the "answer key" debate. In light of what you told us about the tendency of groups to change the answers to match those of the answer key all I can say is that I feel betrayed! My eyes are opened, and I only can wonder how many times in the past I have been led astray by [my trust in authority]. Now I'll be paranoid, and suspicious of all "answers!" I feel like I just found out the truth about Santa Claus.

While certainly I am not proposing that all students need to gain this insight in such a shocking manner, this student has certainly learned an invaluable lesson, one that all mathematics students hopefully will learn. Namely, in mathematics, answers are only important insofar as we can make meaning out of them.

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Enhancing Student Interest in Mathematics using Integrative Curricula

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The purpose of this research was to investigate the effects of two integrative mathematics curricula on high school students' motivation towards mathematics. Both curricula used the environment as the primary theme to create a mathematics course that would be meaningful to the students and actively involve them in the learning process. Meaningfulness and involvement were systematically manipulated to enhance interest in the classroom activities (situational interest) and boost the students' overall interest in mathematics (personal interest). Student survey results from the first curriculum, Mathematics of the Environment, indicated that: (1) student perceptions of meaningfulness and involvement were higher than in the comparison groups, but that (2) overall situational interest was not higher. Structured teacher interviews indicate teachers found the curriculum effective--particularly for depth of cognitive processing. These quantitative data provided the background for designing the Statistics of the Environment course. This curriculum was built using the content-theme of the "local environment" with the action-theme of "student-as-researcher." A strong computer technology component was developed to help students construct and interpret statistical meaning.

This paper reports on the effects of integrative mathematics curricula on students' attitudes towards mathematics. *Integrative* has two implied meanings. First, it is used in its traditional sense to imply a curriculum which uses interdisciplinary themes and concepts. Second, integrative refers to an instructional design orientation that incorporates student motivation to learn, higher order thinking skills, and the basic need for students to learn fundamental knowledge concepts.

The study investigated two curricula developed to enhance student interest in high school mathematics. Both curricula (Mathematics of the Environment and Statistics of the Environment) used the theme of environmental problem-solving as the basis for developing specific student mathematics skills and thinking strategies. These curricula were developed in response to two important papers recommending that mathematics instruction incorporate more "meaningful" mathematics and for high school curriculum to use themes which allow for the development of integrated curriculum (California High School Task Force, 1992; NCTM, 1989).

The purpose of the study was twofold. First, it represented an initial attempt to test the feasibility of systematically enhancing student interest in the mathematics classroom (situational interest). Second, it sought to better explain the relationship between personal interest in mathematics, personal interest in environmental issues, situational interest, mathematical anxiety, and student achievement within the context of these experimental curricula.

The essential problem necessitating the study is that of poor student motivation to learn in the high school mathematics classroom. This lack of motivation can be partially explained by other motivational variables such as the lack of individualistic goal structures, lack of informational feedback, and in optimal challenge (Stipek, 1993). However, the most direct variable for explaining student motivation may be the construct of "interest" (Hidi, 1990). We say this because disinterest in learning is likely the primary manifestation of pervasive classroom boredom (Sarason, 1983). If we better understand the structure of interest, then we may be better able to subsequently enhance student interest through curricula systematically designed to be high in interest for secondary level students.

One of the fundamental distinctions made in the arena of interest research is the difference between personal and situational interest (Hidi, 1990). Personal interest refers to an interest that people *bring to* a situation. It is considered a stable, long-term variable. Unfortunately, personal interest can not be practically manipulated in situations where teachers need to work with a large number of students. Situational interest refers to an interest that people have when *participating in* a situation. It is considered an unstable, short-term variable. While personal and situational interest are distinct constructs, they are related. The educational implications of this distinction are notable. A mathematics classroom that is consistently high in situational interest may significantly alter an individual's personal interest towards mathematics (Hidi & Anderson, 1992). A high personal interest in mathematics may also have the beneficial side-effects of increased academic persistence and achievement (Hidi, 1990).

Conceptually, situational interest is composed of many factors. Initial research indicates that the two strongest factors determining situational interest in the high school mathematics classroom are "meaningfulness" and "involvement" (Mitchell, 1993). Many mathematics classrooms allow students to be passive and uninvolved with content that is meaningless and disconnected from the real world (Davis, 1992; Stodolsky, Salk, & Glaessner, 1991). National and state recommendations have suggested that mathematics instruction needs to embed both meaningfulness and involvement if the mathematics classroom of the future is to be successful (California Mathematics Framework Committee, 1992; NCTM, 1991).

The authors investigated the possibility of enhancing situational interest by systematically designing instruction that: (1) would have a high level of student involvement, and (2) would contain content that many students would consider meaningful. Two integrative curriculum projects were developed and evaluated. The first project curriculum is entitled Mathematics of the Environment (or MOE). This is a 10-12 week, first-year high school curriculum. The course is a "mathematical adventure" in which students learn important mathematical skills and concepts while investigating the environmental health of selected world countries. Students work in groups to propose and present a solution for resolving one environmental problem in their chosen nation. This curriculum was implemented in one public school (16 classrooms total) during the 1993-1994 academic year.

Results from student surveys were used: (a) to test the study's hypotheses, and (b) as formative evaluation information to be used to create a more effective curriculum. There were four hypotheses underlying the study:

1. Students are more interested in environmental issues than in mathematics.
2. We can interest more students in the mathematics class by using an environmental theme.
3. Mathematics curricula can be created which high school students perceive as meaningful.
4. Mathematics curricula can be created which high school students perceive as involving.

Survey results did not support hypothesis 1, but did support hypotheses 2, 3, and 4. Likert survey scales (Mitchell, 1993) were used to measure personal interest in mathematics, personal interest in environmental issues, situational interest in the mathematics classroom, meaningfulness, and involvement in the mathematics classroom. All results were reported on a 1-6 scale with 1 representing strongly disagree, 6 representing strongly agree, and 3.5 representing the mid-point of the scale. For 361 ninth graders surveyed previous to the MOE implementation, personal interest in mathematics (m=3.46, s=1.21) was not significantly different from their personal interest in environmental issues (m=3.57, s=1.11). Thus Hypothesis #1 did not seem tenable.

Interestingly enough, the Pearson correlation between the two kinds of personal interest was 0.15. We defined people as being interested in mathematics (or environmental issues) if their scale score for either variable was greater than 4.5. Using this definition, approximately 21% of the students surveyed could be said to have a personal interest in mathematics. Similarly, 20% of the students had a personal interest in environmental issues. More importantly, due to the low correlation between the two scales, we found that 35% of the students had a personal interest in either mathematics or environmental issues. These results lend some credence to Hypothesis #2 (i.e. that we can interest more students in the mathematics classroom by using an environmental theme). Nonetheless, such results indicate that 65% of ninth grade students have no personal interest in either of these topics.

Student ratings of meaningfulness in the MOE classrooms (m=3.91, s=.91) was statistically significantly higher than in non-MOE classrooms (m=3.62, s=1.09). The resulting effect size difference of approximately .30 is small but important. Similarly, student ratings of involvement in the MOE classrooms (m=3.46, s=.80) were higher than those in non-MOE algebra classrooms (m=3.11, s=1.02). Again, the effect size difference was small but important. These results provide some initial support for Hypotheses 3 and 4. However, by any reasonable criterion standard (rather than by the above reported normative comparisons), meaningfulness and involvement in the MOE classes still needs to be improved.

These survey results helped guide the development of a new, more effective curriculum. The second project curriculum is entitled Statistics of the Environment. This is a twenty week, first-year high school curriculum that also uses the environment as an overarching theme. However Statistics of the Environment is differentiated from Mathematics of the Environment in three keys ways: (1) it contains different and more advanced mathematical content, (2) it focuses on students proposing solutions to *local* rather than national environmental problems, and (3) students learn to become novice researchers by designing a research plan, collecting and analyzing data, and interpreting the practical significance of those analytical results.

From a motivational perspective, the authors sought to more effectively design a mathematics curriculum that students would perceive as meaningful and involving. We thought that a *local* environmental theme (i.e. students studying their own neighborhoods) would be perceived as more meaningful. Second, by incorporating the appropriate and powerful use of computer technology and having students act as novice researchers, we hoped to design a curriculum that students would find more involving.

Computer technology (primarily using the EXCEL spreadsheet software program) was used so that students could construct statistical meaning. EXCEL was used in three ways. First, EXCEL was the basis of forming "number playgrounds" where students could systematically explore specific statistical concepts. Second, EXCEL will allow students to build their own statistical calculators (using only the most basic of the EXCEL functions). Third, students learn through teaching by creating "educational worksheets." (Note: the presentation of this research report will include hands-on examples of each of the curriculum's uses of computers).

In their role as novice researcher, students learn to define a problem, state a hypothesis, create a measurement instrument, and then to collect data in their local community. In these ways students become much more active participants in their own mathematical understanding.

Implementation of *Statistics of the Environment* will begin in the spring of 1995. Structured interviews with MOE teachers indicate that teachers were very enthusiastic about the curriculum, especially in terms of depth of student cognitive processing. Despite the enthusiastic support of teachers who used MOE, however, the authors realized that a teacher training component needs to be built into the Statistics of the Environment curriculum. Traditional teaching practices typically do not lend themselves to curricula which are intended to be highly involving to the student. One of the major cornerstones of the new curricula, therefore, is the effective design of a teacher training program that can prepare teachers for the new curriculum with a reasonable amount of time (one week or less).

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THE EFFECTS OF PSYCHOSOCIAL VARIABLES ON MIDDLE SCHOOL STUDENT PROBLEM-SOLVING ACHIEVEMENT IN MATHEMATICS

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Psychosocial variables that may affect middle school students' confidence with calculators and problem-solving mathematics achievement are explored in this study. Student perceived extrinsic motivation, intrinsic motivation, pride and effort, satisfaction, parental expectations, personal expectations, mathematics anxiety, and social belonging are variables included in this comprehensive model. Descriptive results indicate that students using calculators have high perceptions of these affective factors. Students' initial achievement, parental expectations, extrinsic motivation, and anxiety had significant effects on students' postproblem-solving achievement. Extrinsically motivating students and reducing student anxiety appear to be two important variables that should be addressed.

The use of calculators in mathematics classrooms has been widely advocated by educators and professional organizations (MAA, 1991; MSEB, 1990, 1991; NCTM, 1989, 1991; Willoughby, 1990). Furthermore, the use of calculators has also been found to improve students' mathematics achievement (Bitter & Hatfield, 1992; Hembree, 1990; Hembree & Dessart, 1986). Recently, researchers have begun to relate calculator use to other important educational outcomes like problem-solving achievement and affective areas of learning (McLeod, 1991; Schoenfeld, 1987; Silver, 1985; Williams, Nath, & Waxman, 1994; Nath, Williams, & Waxman, 1994). McLeod found that affective factors varied not only by problem-solving strategy, but also by problem-solving stages such as those identified by Polya (1945): (a) understand the problem, (b) plan solution, (c) carry out the plan, and (d) examine solution. McLeod's findings led Williams, Nath, and Waxman (1994) to look at the effects of student anxiety and calculator confidence on stages of middle school students' problem-solving achievement in mathematics. They found that mathematics anxiety had negative effects on all stages of their problem-solving achievement, while calculator confidence had negligible effects. In a similar study, Nath, Williams, and Waxman (1994) also examined the effects on students' perceptions of social belonging and calculator confidence on students' problem-solving achievement and found that the degree of calculator confidence students reported did not appear to impact students' problem-solving achievement, while students' perceptions of social belonging in their mathematics class did have slight positive effects on each stage of the problem-solving process. Both studies, however, did not include important psychosocial variables like parental attitudes, students' motivation, and students' personal expectations. These psychosocial variables have been found to be critical in creating a supportive learning environment that enhances technology use (Bright, Waxman, & Williams, 1994).

The purpose of the present study was to explore a comprehensive model of psychosocial variables that might affect students' confidence with calculators and to specifically examine factors that influence both students' calculator confidence and stages of problem-solving achievement in mathematics. In the proposed conceptual model, students' extrinsic motivation, students' intrinsic motivation, students' pride and effort, students' satisfaction, student perceived parental expectations, students' personal expectations, students' mathematics anxiety, and student perceived social belonging are exogenous or independent variables that are hypothesized to influence both students' calculator confidence and problem-solving mathematics achievement.

Methods

Subjects

The subjects in the present study were 2,200 sixth, seventh, and eighth graders from five middle schools in a multi-ethnic district located in a major metropolitan city of South Central region of the United States. The ethnic diversity included 30% white, 18% black, 22% Hispanic, and 28% Asian, and of these, 52% were female and 48% were male. Most students were from lower-middle to middle-class homes, and nearly 43% of these students indicated that they spoke a language other than English before they started school.

Instruments

Indicators from three instruments were used to construct the scales and tests for this study Students responded to each item on the scale using a four-point Likert-type scale with "4" indicating strong student agreement ("very true") and "1" indicating strong student disagreement with the item ("not at all true"). The Student Perceived Calculator Confidence Scale was derived from an adapted version of the SUPAI Student Calculator Survey (Bitter & Hatfield, 1992). The scale consists of nine items measuring student attitudes toward the use of calculators. The Student Perceived Mathematics Anxiety Scale (five items), Student Perceived Social Belonging Scale (four items), Student Perceived Intrinsic Motivation Scale (four items), Student Perceived Extrinsic Motivation

Scale (four items), Student Perceived Personal Pride in Effort Scale (6 items), Student Perceived Satisfaction in the Mathematics Classroom Scale (four items), Student Perceived Parental Expectations Scale (four items), and Student Perceived Personal Expectations Scale (3 items) were derived from the MSLQ. All of the scale scores were measured on a four-point scale with "4" indicating strong agreement and "1" indicating strong disagreement.

Students' problem-solving mathematics achievement was measured in October (pre) and again in April (post) using the Four-Step Problem-Solving Test (Hofmann, 1986). The test consists of 10 nonroutine problems each with four related questions. The test is based on a four-step heuristic: (a) read to understand the problem, (b) select a strategy, (c) solve, and (d) review and extend. It is a multiple-choice, paper-and-pencil test designed to measure problem-solving skills of middle school students. The range for the total test is 0 to 40.

Procedures

Each student was issued a calculator for both school and home use along with his or her mathematics text. All teachers of mathematics in this district received 12 hours of inservice training to help incorporate calculators into their instruction. Calculator curriculum activities that had been developed by a writing team during the first year of this project were also made available to each mathematics teacher, and all were encouraged utilize these activities and any others found to aid the teaching and learning of mathematics with calculators.

The Four-Step Problem Solving Test was administered to all students during mathematics classes at the beginning and end of the school year. During both tests, student use of calculators was allowed, although the test is not considered to be calculator sensitive. The student surveys were also administered during the spring. Multiple regression analyses were used to examine the effects of the independent variables and on students' postproblem-solving achievement in mathematics, after statistically controlling for preproblem-solving achievement.

Results and Conclusions

The descriptive results indicate that students using calculators have high perceptions of the affective factors discussed in this study. The scale scores with the highest mean values were (a) satisfaction in mathematics class, (b) personal expectations, (c) extrinsic motivation, and (d) perceived pride and effort. The variables with the lowest mean values were mathematics anxiety and calculator confidence. The standard deviations for most of the variables were quite high with the exceptions of (a) calculator confidence, (b) personal expectations, and (c) perceived pride in effort. The median reliability for all of the individual scales was .67.

Table 1
Means and Standard Deviations for Psychosocial Factors

Scale	*M*	*SD*
Satisfaction in mathematics class	3.53	1.00
Personal expectations	3.44	.51
Extrinsic motivation	3.32	.72
Intrinsic motivation	3.17	.66
Perceived pride and effort	3.24	.51
Mathematics anxiety	2.23	.72
Perceived Social Belonging	3.23	.59
Parental Expectations	3.08	.67
Calculator confidence	2.96	.47

Table 2
Regression of Postproblem-Solving Achievement on Preachievement and Psychosocial Factors

Independent Variables	*Beta*
Preproblem-solving achievement	.60***
Satisfaction in mathematics class	.03*
Mathematics anxiety	-.09***
Personal expectations	.01
Extrinsic motivation	.06**
Intrinsic motivation	.01
Perceived pride and effort	.04
Perceived Social Belonging	.03
Parental Expectations	-.05*
Calculator confidence	-.03
	$R^2=.42$

* $p < .05$
** $p < .01$
*** $p < .001$

The results from the multiple regression analysis indicate that, as expected, preproblem-solving achievement had a statistically significant ($p < .001$) large, positive effect (Beta = .60) on postproblem-solving achievement. Anxiety and Parental Expectations had slight, negative significant effects on postproblem-solving achievement. Extrinsic motivation, however, had a slight, positive, significant effect on problem-solving achievement. The other independent variables had negligible effects on problem-solving achievement.. The independent variables accounted for 42% of the variance on problem-solving achievement.

The results from the present study illustrate the importance of examining psychosocial variables influencing students' problem-solving achievement. Since less than half of the variance in students' calculator confidence and problem-solving achievement was accounted for in the multiple regression equations, there are other areas or variables that still need to be addressed in this exploratory model. Traditionally, computer and calculator usage have been considered effective in the improvement of teaching and learning in mathematics and science (Bitter & Hatfield, 1992; Hembree & Dessart, 1986). Issuing calculators to teachers and their students, however, is no guarantee that successful problem-solving will occur (Hembree & Dessart, 1986). The present study examines some other important areas that need to be considered in order to effectively increase students' problem-solving achievement in mathematics. Extrinsically motivating students and reducing student anxiety appear to be two important variables that should be addressed. The negative effect of parental expectations on achievement needs to be examined in future studies because it is contradictory to many previous studies which have found parental expectations to be beneficial for students.

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JUSTIFYING THE REASONABLENESS OF ANSWERS: PROCESSES OF MIDDLE SCHOOL MATHEMATICS STUDENTS

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This study explored the observable processes and self-reported strategies that middle school students used to determine answer reasonableness. Student justifications were investigated from two perspectives: a) justification of reasonableness when students calculated and analyzed their own solutions and b) justification of reasonableness when students analyzed solutions that were already provided. The findings indicate that the strategy chosen is dependent on whether the student generated the answer or was provided the answer, the mathematical operation involved, and the type and size of numbers in the problem. Students who generated the answer and then provided reasonableness justifications were less global in their thinking and limited in the number of strategies they employed.

Prior to 1980, little attention was given to reasonableness of answer or number sense. Estimation, a component of number sense, received some emphasis beginning in 1977, when the National Council of Supervisors of Mathematics included estimation and approximation as one of the ten basic skill areas in need of development. Three years later, the *Agenda for Action* (NCTM, 1980) called for the regular incorporation of estimation activities in school mathematics. One early study by Reys, Bestgen, Rybolt, and Wyatt (1982) provided insight into how good estimators in grades 7 through 12 estimate. The investigators noted that good estimators shared additional common attributes: (a) had a good grasp of basic facts, place value, and arithmetic properties, (b) were skilled at mental computation, (c) were self-confident, (d) were tolerant of error, and (e) used a variety of strategies within the three key processes (reformulation, translation, compensation). Rubenstein (1985) identified three skills that seemed particularly important to good estimators: (a) were able to multiply and divide by powers of ten, (b) were able to choose the larger or smaller of two numbers, and (c) were able to choose which of two problems will yield a larger value.

Recent documents such as *Everybody Counts* (MSEB, 1989), *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), and

Reshaping School Mathematics (MSEB, 1990) have expanded estimation and emphasized the concepts of number sense and reasonableness of answer as essential to mathematical literacy. These dual, overlapping concepts have been difficult to conceptualize and even harder to analyze from curricular and instructional viewpoints. Howden (1989) stated that number sense could be described as intuitive. Reys (1991) emphasized that although number sense is difficult to define, "number sense is the underlying theme for the study of numbers and thus sense-making is the highest priority as numerical concepts are being constructed by the learner (p. 4, 1991)." Rathmel (1990) proposed a model for number sense that links the idea of reasonableness of answer to three basic components: a) sense of number (meaning, relationships, and magnitude), b) sense of operations (estimation), and c) connections between the mathematical world and real world decisions. Several recent studies (Greeno, 1991; Reys et al., 1991; Markovits & Sowder, 1994) have focused on the ways in which number sense is manifested. According to Sowder (1992), manifestations of number sense include using numbers flexibly when mentally computing, estimating, or determining reasonableness of results.

Objectives of Study

Error analysis experts hypothesize that educators often misinterpret students' reasoning skills because they concentrate on solutions and methods and fail to listen to students' justification for their solutions (Maurer, 1987). The purpose of this study was to explore the self-reported processes that students use to determine the reasonableness of answers. To better understand student reasoning, student justifications were investigated from two different perspectives: a) justification of reasonableness when student calculated and analyzed solution, i.e. student-generated answers, and b) justification of reasonableness when student analyzed solutions that were already provided, i.e. researcher-generated answers.

METHOD

Subjects

Subjects in this study were 240 middle school mathematics students from a large urban/suburban ethnically diverse school district located in a major metropolitan city in the south central region of the United States. The students were divided into two groups. Students in one group (n=120) were given seven computation problems to solve (student-generated answers). In addition, they were asked to briefly explain their processes and to describe the reasonableness of their answers. Students in the second group (n=120) were shown the same seven problems but these problems had answers (some correct and some incorrect) provided (researcher-generated answers). Their perceptions of the correctness of the solution provided and the reasonableness of the answer were elicited through the interview process.

Procedures

Structured individual interviews were conducted over a period of three weeks during the last month of school. The length of each interview was approximately 20-30 minutes and was conducted by trained staff. The seven computation problems submitted to students and used for the interview process were created by the researchers. While the seven problems presented were the same, two different types of questions were asked. The student-generated answers group (n=120) were asked to complete the problems and to explain the processes they used. Students were also asked to revisit each problem, discuss how certain they were of their answer, and explain why they felt their answer was reasonable. The researchers-generated answers group (n=120) were shown the seven problems with answers (some answers were correct and some were not). Students were asked to quickly determine the reasonableness of the given answers and to describe their reasoning.

RESULTS

The descriptive statistics (Tables 1 and 2) revealed several interesting findings about (a) student's understanding of number meaning and mathematical operations for individual problems and (b) strategies used by students to determine reasonableness of answer. The processes used with problems involving fractions and decimals were quite different from those involving whole numbers. In addition, the researcher-generated answers group employed effective strategies with problems involving addition or subtraction more often than with those involving multiplication. The processes used by the two groups seem to be dependent on the type of answer (student-generated or researcher-generated) that was involved. Students who were asked to determine and justify the reasonableness of an answer generated by someone else seemed to respond more globally and use a larger variety of processes than those who worked the problems themselves.

The results in Table 1 indicate that an average of more than 80% of the students who generated their own answers to the seven problems obtained correct answers. A majority of them justified their answers as very reasonable because they had used and trusted the calculator. In contrast, an average of only 67% of the group which attempted to determine the correctness of researcher-generated answers was successful. The highest rates of success for the researcher-generated answers group were for problems #1 (79%), #3 (78%), #5 (75%), and #6 (76%); these four problems involve only addition and subtraction operations. The lowest rates of success were for problems #2 (52%), #4 (59%), and #7 (51%); these problems all involve multiplication. These findings seem to indicate that Rathmel's (1991) broad model for number sense which includes a sense of operation as well as a sense of number may be an important consideration for instruction.

Table 2 indicates that the student-generated answer group predominantly reported that (a) they focused on the size of the answer when determining reasonableness or (b) they could not determine reasonableness. Almost 70 percent of the students in this group used calculators to find their answers, with most of the remaining students using paper

TABLE 2

Percentage of Effective Self-Reported Strategies Used by Middle School Students to Determine Reasonableness of Researcher-Generated Answers (n=120) and Student-Generated Answers (n=120)

	Problem Number						
Self-Reported Processes	1	2	3	4	5	6	7
Evaluated Part of Answer							
Researcher-Generated	.20	.07	.15	.57	.35	.00	.15
Student-Generated	.03	.00	.02	.04	.06	.04	.05
Looked at Format of Answer							
Researcher-Generated	.05	.20	.03	.02	.02	.00	.00
Student-Generated	.05	.00	.02	.00	.01	.00	.00
Focused on Size of Answer							
Researcher-Generated	.00	.07	.00	.19	.13	.65	.33
Student-Generated	.27	.11	.43	.54	.24	.39	.49
Considered Whether Answer Seemed Good/Bad							
Researcher-Generated	.06	.05	.17	.03	.02	.03	.00
Student-Generated	.06	.15	.05	.08	.03	.03	.06
Used Front End Estimation							
Researcher-Generated	.12	.00	.22	.03	.09	.21	.11
Student-Generated	.21	.00	.03	.01	.18	.04	.00
Guessed							
Researcher-Generated	.04	.10	.06	.14	.31	.00	.39
Student-Generated	.00	.01	.00	.01	.01	.00	.04
Calculated Exact Answer Mentally							
Researcher-Generated	.47	.47	.27	.00	.00	.09	.00
Student-Generated	.19	.27	.12	.04	.15	.25	.05
Visualized Problem/Answer							
Researcher-Generated	.00	.00	.00	.00	.00	.00	.00
Student-Generated	.00	.09	.00	.00	.00	.00	.00
Could Not Determine Reasonableness							
Researcher-Generated	.06	.04	.10	.02	.08	.02	.02
Student-Generated	.19	.37	.33	.28	.32	.25	.31

Educational Significance of Study

Spangler (1992) reported that the assessment of students' beliefs about mathematics can help teachers plan instruction and structure the classroom environment to create changes. The results of this study indicate that the processes students used to determine reasonable answers were often algorithmically based and seldom based on conceptual understanding or number sense. Reys (1991) advocates, instruction that focuses on purposeful activities which require students to think about numbers, their relationships, and their connections to quantitative information seen in everyday life. The findings from this study support that plan.

TABLE 1

Percentage of Correct Responses of Middle School Students for Problems with Researcher-Generated Answers (n=120) and Student-Generated Answers (n=120)

Problem Number	Problem (Answer Generated by Researcher)	Percentage Correct Researcher-Generated Answers	 Student-Generated Answers
1	Subtract 4.03 from 16.86 (12.83)	.79	.78
2	$\frac{2}{3} \times 3 =$ $\left(\frac{6}{9}\right)$	.52	.83
3	243.61 - 57.24 (186.37)	.78	.83
4	596 x 247 (14573)	.59	.81
5	$3\frac{3}{4} + 9.2 =$ (17.45)	.75	.77
6	348 + 999 = (447)	.76	.91
7	(2,463 + 2,463) x 50 = (123613)	.51	.81

and pencil to do their calculations. These students expressed that if a problem is done with a calculator OR if they are very sure of their work, a determination of reasonableness cannot or should not be done. In contrast, the researcher-generated answers group used a much greater variety of strategies which seemed dependent on the type of problem and the size of the numbers involved. The majority of these students reported that for problems #1 and 3 which involved subtraction and #2 which involved small numbers they calculated an exact answer mentally in order to determine reasonableness, for problem #4 which involved multiplication and #5 which involved the addition of a fraction and a decimal they evaluated only part of the answer, for problem #6 they focused on the size of answer, and for problem #7 they guessed. For each problem, the two most reported strategies used by the student-generated answers group were never the same as the two most reported by the researcher-generated answers group.

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VOICING THE VOICELESS: A MATHEMATICS TEACHER'S AND AN STUDENT'S VIEWS AND BELIEFS ABOUT TEACHING AND LEARNING MATHEMATICS

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The purpose of this paper is to report the views and beliefs of a practicing secondary school mathematics teacher, Sofia, and one of her students, Sam, about teaching and learning mathematics. Learning about their views and beliefs has helped the researchers to make sense of the nature of mathematics classroom practices. This report is part of a wider research conducted in the context of Costa Rica, under a constructivist framework (von Glaserfeld, 1987).

Sofia's practices of course have a thorough meaning only when looking at her overall practice. It was found that Sofia's beliefs-in-practice about teaching and learning mathematics influenced the way she conceptualized her role as a teacher in the classroom. She viewed the role of the teacher essentially as prescribing to students the mathematical concepts so that what they needed to do was to practice what the teacher had prescribed in class. This belief was almost contradictory with her stated belief.

Sam had constructed two different views of the student's role learning mathematics. In a mathematics classroom, it was to follow the teacher's way to approach mathematical activities with the intention of showing the teacher that he was proficient in mathematics. A second view, when he was engaged doing mathematics, he could approach it his own way, but in a manner that made sense to him.

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STUDENT AUTHORITY IN THE CLASSROOM

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The results of recent studies indicate that female and minority students perform consistently below white students on mathematics achievement tests. In addition, both female and minority students take fewer mathematics courses and are greatly underrepresented in science and technological careers. At the same time, these populations are enrolling in colleges and universities in larger numbers than ever before. Many enter college with quite limited mathematical skills.

Cultural characteristics may interfere with students' learning of mathematics. Native American and Hispanic women are often discouraged in mathematics and math related careers. As individuals explore and internalize their world, they do so in a unique relationship to their own culture and society. This exploration and internalization leads individuals to develop a set of beliefs and values, which cannot be isolated from the culture in which they were derived. It was the intent of this project to examine students' perspectives of mathematics and their use of discourse to suggest the existence of stages of mathematical development based on the positions set forth by Perry (1970) and the perspectives of Belenky et al (1986), as stated in their book, Women's Ways of Knowing.

Thirteen students who tested into a basic arithmetic course participated in this study. Each class session of the course was video-taped. All students in the class participated in a series of three interviews. The communications that occurred within the classroom demonstrate the importance of negotiation in the learning of mathematics. Discourse occurring in the small groups allowed the students the opportunity to construct mathematical structures, build on each others' work, and negotiate mathematical meaning, while the whole class discussions permitted students to formalize their arguments and reflect on the arguments of others.

The analysis of the tapes focused on various types of communication including: in-depth mathematical discussions between and among students, and between students and teacher; negotiation of mathematical meaning and the use of mathematical procedures; processes used by students to solve mathematical problems; beliefs about the nature of mathematics; and how students use and develop authority in the learning of mathematics.

Belenky, M., Clinchy, B., Goldberger, N., and Tarule, J. (1986). Women's Ways of Knowing. New York: Basic Books. Perry, W.G. (1970). Forms of Intellectual and Ethical Development in the College Years. New York: Holt, Rinehart and Winston.

When do the equilateral triangles and the congruency let to be boring? Introduce to Deltahedral Models.

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Abstract

This poster report a technic to built Deltahedral Models, which are polyhedrons with equilateral triangles by faces. Also, it propouses several exercices and some actual reserches in Discrete Geometry. The utility in the clasroom of this experience is been analized.

The teachers and students found a difficulties in the study of congruency with flat figures: the formal proofs of theorems, the concatenation of definitions, the comprehension of problems. Furthermore, the tridimensional perceptions are not develop in the classroom, the constructive habilities are unattended; we do not inform them about the actual mathematic researches. The recourse we are presenting must be a first experimental and constructive approach to the congruency concept. We could add other concepts too, like simetry, translation or ciclic groups.

Folding trips of paper

The propouse is the construction of Deltahedral Models based on equilateral triangles aligned in rectangular trips of paper. It applies the congruency of segments and angles. We can interpret this as the SAS criterion of triangles' congruency, with equilateral triangles aligned in a rectangular trip. Convention to folding trips of paper:

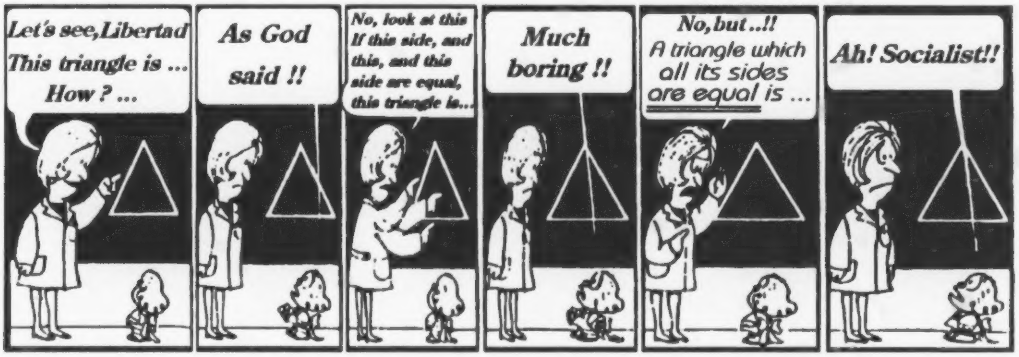
-1 = Fold in mountain (Fold hide the drawing).
0 = No fold, i. e., two triangles in the same plane.
+1 = Fold in valley (Fold show the drawing).

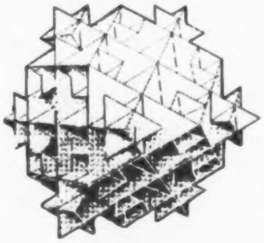
Ensambled of Deltahedral Models

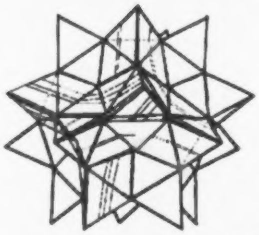
The conditions for the built of polyhedral models are two:

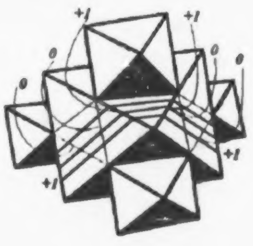
1) Each arista is crossed at least one time by someone trip of paper; in other words, any arista is not open slit.

2) If all trips are coloured in diferents tones, then the visible areas of each tone are equals.


Let's see,Libertad This triangle is ... How ? ...
As God said !!
No, look at this If this side, and this, and this side are equal, this triangle is...
Much boring !!
No, but..!! A triangle which all its sides are equal is ...
Ah! Socialist!!







Teachers' Beliefs and Attitudes

RESEARCH PAPERS

A Teacher's Perception Of Time In A Mathematics Classroom: A Case Study Of Reflection On Reflection-In-Action 155
Kathy M. C. Ivey

Preservice Secondary Teachers' Beliefs About Mathematics And Their Expectations About Student Performance On Open-Ended Assessment Tasks 162
Barbara W. Grover and Patricia Ann Kenney

Changing The Mathematics Learning Environment In Relation To Beliefs, Knowledge, And Practices 169
Patricia A. Jaberg and Cheryl A. Lubinski

Connecting Orientation Towards Authority To First-Year Teachers' Thinking About Teaching 176
Barry E. Shealy

ORAL REPORTS

A First Year Secondary Mathematics Teacher's Orientation Toward Authority: Implications For Professional Development 183
Bridget Arvold and Thomas J. Cooney

POSTERS

Emerging And Evolving Themes: Case Studies Of Three Preservice Secondary Mathematics Teachers' Beliefs 184
Roger Day

Warning-Hard Hat Area: Falling Beliefs 185
Annette Ricks Leitze

A TEACHER'S PERCEPTION OF TIME IN A MATHEMATICS CLASSROOM: A CASE STUDY OF REFLECTION ON REFLECTION-IN-ACTION[1]

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This study is a systematic attempt to explain the varying sense of time demonstrated by one teacher and to explore some implications of his time sense. The method used, stimulated recall with videotapes, is a fruitful way of studying the complex activity of teaching from an emotionally safe distance. Results indicate that there are important differences in the teacher's sense of time when it is intuitive and part of everyday teaching practice, when it is questioned within the context of teaching, and when it is considered in the abstract as part of a teaching dilemma. These distinctions reveal subtleties in classroom interactions that can have dramatic consequences. Furthermore, the types of reflection resulting from this method engender deeper understanding of the classroom for both the teacher and the researcher.

What does time mean in the context of the classroom? How does the teacher's sense of time affect pedagogical decisions? These questions are particularly compelling in light of Romberg and Pittman's (1990) contention that "reform implies reallocation of classroom time to new organizational structures and student activities" (p. 224). Appealing to an internal sense of time is one way that teachers determine time allocation during a lesson. Time sense, however, is subtle and does not usually intrude on conscious awareness, which makes investigation of it difficult. In fact, Slife (1993) points out that "time's quality in relation to a subject of interest" has rarely been investigated (p. 94). One way of exploring subtle, transparent views is alluded to by Kilpatrick (1985): "in reflecting on our experience, we move out of the plane of our everyday existence. We give meaning to experience by getting outside the system" (p. 6). Building on these observations, this paper has two purposes. First, it suggests a framework for discussing a teacher's varying sense of time and multiple meanings for time which acknowledges the subjective nature of time sense. Second, it illustrates the value of a particular method for investigating a teacher's reflections on the teaching experience by exploring some reflections on the use of time and the meaning of time in a particular classroom episode.

Methodology

The data reported in this study were gathered as part of a larger project. Using the technique of stimulated recall, Mr. Scott (the cooperating teacher) and I reviewed videotapes which had been made the

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previous year in one of his eighth-grade beginning algebra classes as part of the larger project. The resulting discussions were audio-taped and transcribed. These transcripts, the videotapes that were viewed, and fieldnotes from classroom observations form the corpus of data used here. In particular, one videotape of group work is the focus of the analysis reported here. Names of all participants are pseudonyms.

Analytic Framework

Slife (1993) contends that "the use and perception of time depends upon the context in which it occurs" (p. 220). To examine contextual based differences in a teacher's time sense, Heidegger's three modes of engagement are useful: *ready-to-hand*, *unready-to-hand*, and *present-at-hand* (see Packer, 1985, for more details). Basically, ready-to-hand describes an object which is being used in an ongoing activity without conscious consideration of the object. Unready-to-hand describes objects which are being consciously considered in the context of an ongoing activity. Objects are present-at-hand when they are being considered abstractly, separate from an activity. These three concepts have also been used to describe actions of a teacher while teaching (Williams, 1993). This framework is used here to discuss Mr. Scott's varying sense of time.

Also important to this analysis, Schon's (1983, 1987) work on reflective practice informs the method used in this study and helps to clarify Mr. Scott's reflections on the meaning of time in the classroom. Schon distinguishes two types of reflection: *reflection-in-action* and *reflection-on-action*. Reflection-on-action, the more familiar form, is logical, rational, deliberate consideration of a matter. Reflection-in-action, Schon's other type, is a process in which one "sees" information or actions in a new way and through this *reframing* is able to understand in a new way. The distinction between reflection-on-action and reflection-in-action in combination with different modes of engagement is used to explore Mr. Scott's reframing of the meaning of time in the classroom during a particular lesson.

Analysis

Time had a strong influence on Mr. Scott's decisions about what to include in a lesson. During one of our conversations, he underscored the importance of time in his decisions about what to teach.

> Mr. Scott: Usually I teach to the assignment, what they [students] are going to be doing that night. . . . A lot of times I spend most of the time making sure that the skill's adequate, and I

> don't have it [time to do more than skill-building], you know. A lot of it is quickly down to time, the time problem.

Mr. Scott's sense of time played important and varied roles in his teaching. During direct participation in the classroom, time was ready-to-hand for him. The small clock ticking in the back of his mind was ever present but not obtrusive. Later, with the clock turned off, a different view of the classroom emerged. While we viewed the videotapes, time became unready-to-hand, requiring direct attention; time also became present-at-hand as the subject of consideration. Changes in the modes of engagement with time led to a deeper understanding of the interactions in this classroom.

Viewing videotapes of one's own teaching allows for careful consideration of actions in a logical, somewhat detached manner, that is, reflection-on-action. Using the method of stimulated recall was intended to prompt Mr. Scott's reflection on his teaching in this class. Somewhat unexpectedly, viewing videotapes also allowed for reflection-in-action by providing him a different orientation, through different modes of engagement, from which to understand the interactions in the classroom. This method of stimulating reframing of classroom events has intuitive appeal for exploring the teaching experience from the teacher's perspective. The following example involving the meaning of time in the classroom substantiates the value of this form of reflection. In this example Mr. Scott's ready-to-hand time sense played a crucial role in his pedagogical decisions while in the class, and different modes of engagement with time played equally important roles in his reframing of those same events as we viewed the videotape later.

During this classroom episode, groups of three students were working on a fair apportionment problem based on ballots which ranked students' preferences among three flavors of ice-cream. The groups were asked to describe three *fair* methods for choosing one flavor, to decide which of their methods was *fairest*, and to justify their choice. One group of three--Lisa, Kelly, and Nancy--quickly devised three methods, rejected one method, and spent about thirty minutes arguing over which of the other two was fairest. Nancy advocated one method, Lisa advocated the other method, and Kelly remained undeclared. Toward the end of the period, Mr. Scott checked on this group. He asked what methods they had come up with and which they thought was fairest. The girls explained their disagreement, while Mr. Scott listened to each one in turn. He then tried to help the group out of its dilemma and still allow each girl to maintain her own opinion. Unfortunately this did not bring about an immediate resolution, and after

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a short exchange between group members, Mr. Scott said, "OK, what I would do is just pick one, 'cause you need to zip it off." This statement resulted in Nancy's complete capitulation, and the group had an answer. They did not discuss the rest of the problem directly, talking only enough to ensure that their three papers would be similar.

Mr. Scott's sense of time was ready-to-hand during this interaction and led him to encourage this group to choose an answer by mentioning the time. For Mr. Scott, the amount of time remaining in the class was an implicit cue for when to move on to another activity. His intention seems to have been to break an impasse so that the group could continue working. Time, here, was a face-saving device. In this case, he offered lack of time as a reason for one student to give in without giving up her personal opinion. In fact, Nancy acknowledged this justification soon after Mr. Scott left the group. In response to the worksheet question of why the chosen method was fairest, Nancy said, "We choose number two [their second method], just because our group says so!"

His decision to remind the group of the amount of remaining time also resulted in a cessation of real discussion for the rest of the worksheet. When we viewed this videotape, Mr. Scott experienced reflection-in-action about how this group worked together and about the results of his comment about time. Being able to watch the group's interaction without the distractions of nine other groups and without changing that interaction by his presence gave Mr. Scott a different view of this group. Here time became unready-to-hand. He was still in the context of the classroom through the videotape, but his sense of time had altered, so that the meaning *to the students* of his comment about time became the focus of his attention.

> Mr. Scott: This is a little unfortunate, and it's partly due to me kind of saying, "OK, you need to start wrapping it up." They're just all [acting like], "OK, let's get the answers down, and not talk about it anymore."

Mr. Scott reframed these events and now saw that his comment had spurred the students to change their group's work from discussion to production, rather than spurring them to move past their disagreement and continue working as they had been. This type of reflection was dependent upon his chance to change his mode of engagement with time by focusing on the meaning of his comment about time, and upon the opportunity of viewing the work of this group before and after he interacted with them.

Mr. Scott was very concerned about the effect of his comment. He returned to this issue later and introduced a dilemma that the issue of time raised. Now time became present-at-hand. It was the focus of his comments, but in an abstract sense. He was able to consider time in a more general way through his reflection on the earlier reflection-in-action.

> Mr. Scott: So you get this huge discrepancy between how long it's going to take groups, as far as managing a class of ten groups, it's--it's difficult to have a two-day thing.... If you cut it off--you know, either--if you extend it, you're going to end up with a whole bunch of groups doing nothing. If you shorten it, what you're going to end up--and even maybe just give one or two groups an extension, even that--you get--you give the message out that it should have been done by now, you messed up because you're not done, and that's--that's a difficult obstacle to get around.

At this point, Mr. Scott had reframed his view of this group's work and the effects of his brief interaction with them. Reflection on this new way of viewing their work led him to articulate a common dilemma in managing group work--how to keep everybody involved, while encouraging exploration and communication.

As Mr. Scott talked about this episode, his time sense seemed different. In the classroom, the amount of remaining time seemed to be in the background, that is, ready-to-hand. His original comment to the group about time was, in his mind, not really about time, but rather was a way to refocus the work of the group onto a different problem. When he was out of the classroom, time took on a different aspect. Time became unready-to-hand, and he now saw it as a goad to finish the assignment, much the way the students had in the classroom. With this reframing of the meaning of his comment, the appropriateness of the comment was called into question. Further reflection on the message of his comment led him to consider a dilemma involving time as present-at-hand, with time taking on an evaluative role.

Our discussion, intended to encourage more reflection on this new view of the classroom, revealed Mr. Scott's reassessment of this pedagogical decision.

> Mr. Scott: The main thing is I wouldn't kill that [discussion], and give them the message that it's more important to get it done than to talk about it.

Clearly, Mr. Scott had reassessed his decision in the classroom to remind the group of the amount of remaining time and of the need for them to finish.

Discussion

This study resulted in two valuable conclusions. First, a teacher's sense of time varies dramatically and can have great influence on his or her teaching. For most teachers, a sense of the amount of time remaining in the class and in the school year is ever present and is influential in guiding their actions in the classroom. Because teachers do have goals for particular lessons, a ready-to-hand time sense can help to determine whether and how those goals are met. A teacher's time sense, in the ready-to-hand mode, may serve to hinder exploration in the classroom, as it did in the episode discussed earlier. Time, in the unready-to-hand mode, may be the basis for a new way of viewing interactions in the classroom. In the episode discussed above, Mr. Scott reframed his comment about time and subsequently judged his interaction with Lisa, Nancy, and Kelly to be counterproductive. A present-at-hand sense of time can help to focus discussions on complex teaching dilemmas, such as how to manage group work so that each group has enough work and enough time. This issue was particularly troublesome for Mr. Scott, and is likely to recur generally as teachers try to increase collaborative work in their mathematics classrooms.

The ready-to-hand time sense that has served teachers well in the past may need to be examined. The example discussed here gives a clear indication of how time sense can directly affect the pedagogical decisions of the teacher. Different modes of engagement with time can result in different meanings for time and different responses to time, as well as different types of reflection.

The second point is the value of the different type of reflection on teaching which is possible when a teacher can view himself or herself in the act of teaching. The use of videotapes of his own teaching gave Mr. Scott the opportunity to view his own work from an emotionally safe distance--the events had already transpired; the class was over. Certainly the intent of stimulated recall used in this way is not to make a teacher constantly conscious of every choice and every action while teaching. To engage in a complex activity like teaching requires that some, or even most, of the actions be ready-to-hand. The real benefit is the chance to switch modes of engagement and to experience reflection-in-action, thereby gaining the general insight that there are multiple ways of viewing actions and information. In Mr. Scott's case, this opportunity allowed him to see how a group interacted and the results of his interaction with them, without the attendant responsibilities of managing the classroom. He was able to reframe events and to reconsider some of his own actions. In doing so, he was able to articulate and explore a difficult issue relating to time in the class.

For Mr. Scott, reviewing tapes of his own class allowed for both reflection-on-action and reflection-in-action, through different modes of engagement with teaching. For me as a researcher, this method provided a new way of examining teaching. In a sense, we could "pause" the class, discuss teacher intent, and then continue the class without interfering with instruction. Even though these discussions were necessarily reconstructions, they were more firmly situated in the event than would have been possible through simple recall. In short, this technique added a new dimension to the picture of the teacher's experience of the teaching act. The new perspectives gained by both teacher and researcher led to new and more complete ways of seeing the classroom.

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PRESERVICE SECONDARY TEACHERS' BELIEFS ABOUT MATHEMATICS AND THEIR EXPECTATIONS ABOUT STUDENT PERFORMANCE ON OPEN-ENDED ASSESSMENT TASKS

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Data were obtained from 15 preservice secondary teachers, who responded to two instruments (a Beliefs Inventory and a Scoring Activity) at three points in time: at the beginning of a Mathematics Teaching Methods course, at the end of the course, and toward the end of the practice teaching experience. The results were analyzed for changes in the teachers' views over time using two categories, Dualism and Relativism, (from Perry's [1970] model) and changes in criteria for evaluating student work. The major findings included: 1) the preservice teachers' beliefs were aligned with a Relativist world view across all time periods; 2) assessment practices were consistent with a Relativist view; 3) changes from holistic to analytic scoring accounted for changes in score levels assigned to student work.; and 4) criteria such as effort, conceptual understanding and adequate explanations influenced the assignment of score levels.

The NCTM Standards documents (1989, 1991) and the draft version of the Assessment Standards (1993) call for important changes in the way mathematics is taught and assessed. An underlying assumption of all these standards-creating efforts is that teachers, as the primary assessors of student achievement, should have the beliefs, knowledge, and skills necessary to assess important content, properly interpret student work, and draw valid inferences from that work. We are just beginning to learn about what criteria teachers consider when evaluating students and what relationship those criteria have to the teachers' view of mathematics (Wilson, Cooney, & Badger, 1991).

In the current study, we analyzed beliefs about mathematics teaching and learning as well as assessment using two categories, Dualism and Relativism, from Perry's (1970) theory about how individuals view their worlds. With respect to mathematics education, the Dualist view aligns with a view that mathematics involves a set of prescribed rules and procedures that need to be memorized, that the teacher's role is to know the "truth" (correct answer or procedure) and convey it to the student, and that the student's role is to passively accept the knowledge the teacher conveys. The Relativist view aligns with a view that mathematics is a dynamic and evolving discipline, that the teacher's role is to establish a learning environment in which the student is an active participant in making sense of the concepts and ideas to be learned. We also analyzed changes in beliefs on individual items on a beliefs inventory and changes in score levels for samples of student work. We were also interested in the extent to which assessment practices reflected beliefs.

The study addressed two research questions:

1. Using as a framework two categories, Dualism and Relativism, from Perry's (1970) theory about how individuals view their worlds, what changes occur in preservice secondary teachers' beliefs about mathematics over the course of an academic year in which they have been enrolled in a teaching methods class and done practice teaching?

2. To what extent do preservice secondary mathematics teachers' criteria for evaluating student work change over the course of an academic year in which they were enrolled in a teaching methods class and done practice teaching?

Method

Subjects

The subjects (hereafter called "teachers") were 15 preservice secondary school teachers enrolled in a one-semester, six-credit Mathematics Teaching Methods course at a major urban university in the northeast. The course emphasized an NCTM Standards approach to teaching and learning mathematics. Nine teachers were participating in a 13-month program that would result in a Master of Arts in Teaching (MAT) degree, and were engaged in practice teaching at the time they were enrolled in the methods course. The other six teachers were completing their fifth year of undergraduate work to obtain a Bachelor of Arts (BA) degree and certification to teach secondary school mathematics, and engaged in practice teaching after completing the methods course. While enrolled in the methods course, the six teachers spent one day per week observing the classes of the master teacher under whose supervision they would do their practice teaching. Practice teaching experiences varied with the individual because of degree program (MAT vs. BA), assignment to different schools (urban, suburban), grade levels (7-12), and supervising teachers.

Instruments

Each of the following instruments was administered at the beginning (Fall) and the end (Winter) of the Methods course and at the end of the practice teaching experience (Spring).

Beliefs Inventory[1]. The Beliefs Inventory consisted of 34 statements related to the teaching and learning of mathematics. Twenty-four statements, seven of which addressed assessment practices, were deemed suitable for reflecting a Dualist or Relativist perspective[2] and for 15 of the 24 statements, the teachers provided brief rationales for the selected choice.

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[1] The Beliefs Inventory used in this study is an augmented version of one used by the QUASAR (Quantitative Understanding: Amplifying Student Achievement and Reasoning) Project, a Ford Foundation sponsored project designed to enhance mathematics instruction in middle schools with high percentages of students from economically disadvantaged communities. The three assessment tasks, which have been released for public dissemination, were developed for use in the Assessment component of QUASAR. A summary of the QUASAR Project and associated papers can be obtained by writing to: Dr. Edward A. Silver, Director, QUASAR, LRDC, 3939 O'Hara Street, Pittsburgh, PA 15260.

Scoring activity. The scoring activity involved three open-ended mathematics tasks, with each task accompanied by seven or eight sample responses that had been generated by middle school students. Using a range of score levels from 0 to 4, the preservice teachers evaluated each student response, providing a brief rationale for the score level selected. For each task they also wrote descriptions of their criteria for a full-credit response (score level 4).

Results

Research Question #1: Changes in Beliefs

The responses of each teacher to each of the 24 statements from the Beliefs Inventory at each of the three time periods were coded as reflecting a Relativist view, Dualist view, or "unsure", depending upon the teacher's selection on the Likert scale. In a given time period, a teacher was classified as having a Relativist or Dualist point of view, if the responses to three-fourths of the statements (i.e., 18) reflected that view. All preservice teachers were classified as having a Relativist point of view at every time period. The proportion of responses in which teachers changed their opinion (i.e., Relativist or unsure to Dualist; Dualist or unsure to Relativist; Relativist or Dualist to unsure) remained consistent from Fall to Winter (23%), from Winter to Spring (20%), and from Fall to Spring (26%)[3], with no significant differences in the percentages.

Beliefs about assessment. Analysis of the changes in beliefs on the seven statements about assessment practices resulted in significant differences ($X^2 = 6.22$, df = 2, $p < .05$). A greater number of changes in responses to the assessment statements occurred from Fall to Winter (21%) and from Fall to Spring (24%) than from Winter to Spring (12%). The reason for this drop in the number of changes in responses from Winter to Spring is unclear.

During each time period, the majority of the changes in the assessment statements involved shifts from Dualist or unsure to a Relativist perspective. Across all time periods, 56% of the changes on average shifted to Relativist, 15% shifted to Dualist, and 29% shifted to unsure. This means that although all of the preservice teachers maintained an overall Relativist position throughout the academic year, their beliefs shifted even more completely to a Relativist position.

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[2] Two of the statements were eliminated from analysis because their ambiguous wording did not clearly address the issues and led to confusion for the teachers. Eight statements were eliminated from analysis because they could not be classified as reflecting a Dualist or relativist view.

[3] In addition to identical responses from one time period to the next, a shift from Agree to Strongly Agree or the reverse and from Disagree to Strongly Disagree or the reverse were considered to be no change.

On five of the seven assessment statements, 80% or more of the preservice teachers' responses reflected a Relativist view for all three time periods. On the remaining two statements, #12 - *Written tests are the best means of assessing student progress in mathematics class* and #33 - *Proper use of the English language should be given equal weight to mathematical understanding when evaluating students' written solutions to math problems,* there is less consensus. On #12, for all three time periods, and on #33, in the Winter and Spring, about half the teachers disagreed with the statement (a Relativist perspective), about one-fourth agreed (a Dualist perspective), and about one-fourth were unsure. On #33, in the fall, two-thirds of the teachers disagreed with the statement, one-third agreed, and no one responded unsure.

Nearly all teachers who disagreed with or were unsure about statement #12 supported their choice by saying that there was no "best method" and that multiple forms of assessment should be implemented to accommodate diversity among students and to assess a variety of skills. Teachers who agreed supported their choice by commenting on the value of expressing ideas in written form, but also acknowledged the limitations of written tests. Those who agreed with statement #33 believed that proper use of the English language was an important workplace skill and promoted interdisciplinary learning, while those who disagreed or were unsure believed that, although use of the English language was important, it should not be given "equal weight" with mathematical understanding. None of the students included correct use of the English language in their criteria for scoring student responses, however.

Research Question #2: Changes in Score Levels and Criteria for Evaluating Student Work

Patterns of score-level changes for each task. For each of the three tasks, the patterns of score-level changes for the group of teachers were analyzed using a nine-cell table comparing the three scoring occasions [Fall-Winter; Winter-Spring; Fall-Spring] and the three score level change directions [higher, lower, no change]. The findings showed no significant chi-square values, suggesting that, as a group, the teachers did not show a change in their scoring patterns across scoring occasions for any of the tasks. This lack of change parallels the lack of change in patterns of responses on the beliefs inventory.

A second analysis investigated each teacher's pattern of score level changes for each of the tasks. Two teachers, Teacher A and Teacher B, showed significantly different scoring patterns between scoring occasions for the Decimal Task (sign test, $p < .05$), shown in Figure 1.

Figure 1. Decimal Task *Circle the number that has the greatest value.*
.08 .8 .080 .008000
Explain your answer

Teacher A assigned lower score levels in the Winter than in the Fall and higher score levels in the Spring than in the Winter. A plausible explanation for this pattern emerged when this teacher's criteria for a full-credit response were analyzed for the three scoring occasions. In her Fall criteria, Teacher A appeared to be evaluating in a holistic manner. However, in both the Winter criteria (1 pt Answer; 3 pts. Understanding) and the Spring criteria (2 pts. each for Answer and Understanding), she shifted to a more analytic scheme. This suggests that the lower score levels in Winter can be attributed to the shift from holistic to analytic scoring, and the higher Spring score levels to the increase in "points" assigned to a correct answer.

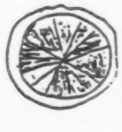
Teacher B assigned score levels that were lower in the Spring than in either the Fall or Winter. Like Teacher A, Teacher B switched from a holistic scoring scheme in the Fall to a "point" based analytic scheme in the Winter, but his scheme was weighted in favor of the answer ("Correct answer-3; adequate explanation-1"). For the Spring scoring occasion, Teacher B refined his scheme to give equal weight (i.e., 2 "points" each) to answer and explanation.

Patterns of score-level changes for each work sample. Looking at each student work sample separately, it was found that teachers significantly changed their score levels for the Fall-Spring comparison (sign test, $p < .05$) for three work samples. For work sample I, shown in Figure 2, score levels were significantly lower in the Spring than in the Fall, with the majority of changes from score level 4 (Fall) to score level 3 (Spring). The reason given most frequently for the Spring scoring involved the fact that the other decimals were not also compared with the "one whole".

Figure 2. Work Sample I (Decimal Task)

Circle the number that has the greatest value.
.08 .8 .080 .008000

Explain your answer



$\frac{8}{10}$ of the pie
is the nearest to
a whole number 1

For work sample II, shown in Figure 3, score levels were significantly higher in the Spring than in the Fall, with most of the changes from score level 0 (Fall) to score level 1 (Spring). An analysis of the justifications that teachers provided showed that the most common reason involved giving credit for effort.

Figure 3. Work Sample II (Number Theory Task)

Yolanda was telling her brother Damian about what she did in math class.

Yolanda said, "Damian, I used blocks in my math class today. When I grouped the blocks in groups of 2, I had 1 block left over. When I grouped the blocks in groups of 3, I had 1 block left over. And when I grouped the blocks in groups of 4, I still had 1 block left over." Damian asked, "How many blocks did you have?"

What was Yolanda's answer to her brother's question? Show your work.

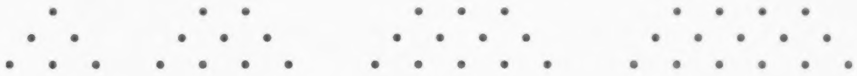
$2+3+4+1+1+1=12$

Answer: 12

For work sample III (Pattern Task), shown in Figure 4, score levels were also significantly higher in the Spring than the Fall, with the majority of changes from score level 2 (Fall) to score level 3 (Spring). These changes were based on the teachers' observation that although the student's pattern for the fifth figure was correct, the explanation was inadequate.

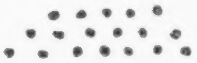
Figure 4. Work Sample III (Pattern Task)

For homework Miguel's teacher asked him to look at the pattern below and draw the figure that should come next.



Miguel does not know how to find the next figure.

A. Draw the next figure for Miguel.



B. Write a description for Miguel telling him how you knew which figure comes next. →

5
6
7

The decrease in weight given to correct answers and the increase in weight given to effort, understanding, and adequate explanations are consistent with the teachers' Relativist view.

Discussion

These results extended our previous findings (Grover & Kenney, 1993) in several ways: 1) the preservice teachers' beliefs about the nature of mathematics and the teaching and learning of mathematics were consistent with their beliefs about assessment practices and were aligned with a Relativist view; 2) their beliefs were consistent not only prior to and following their experiences in the methods course but also following the practice teaching experience; 3) assessment practices were consistent with a Relativist view; 4) changes from holistic to analytic scoring approaches rather than changes in criteria may account for changes in score levels assigned to student work; and 5) in addition to the correct answer, criteria such as effort, conceptual understanding, and adequate explanations influenced the assignment of score levels.

These results contrast with those of an earlier study by Wilson, Cooney, and Badger (1991). Wilson and colleagues found that experienced teachers held a Dualist world view, that their "evaluation practices reflect a narrow view of what constitutes mathematical understanding " (p. 15), and that they "would hesitate to use open-ended items in testing situations" (p. 21). The preservice teachers in this study held a Relativist view throughout the year, seemed comfortable evaluating open-ended items, and used multiple criteria to determine score levels. The differences in time periods of the studies, one year vs. three years after publication of the NCTM Standards, and the differences in the opportunities available to preservice teachers compared to veteran teachers to become acquainted with the NCTM standards may be contributing factors. Further research is warranted to explain the contrasts.

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CHANGING THE MATHEMATICS LEARNING ENVIRONMENT IN RELATION TO BELIEFS, KNOWLEDGE, AND PRACTICES

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The purpose of this study was to analyze K-4 elementary teachers' beliefs and instructional practices both before and after they have had access to knowledge about children's mathematical thinking. All eleven teachers from one elementary school and their principal volunteered to attend 12 inservice sessions[1] *involving 34 hours over a nine month period. Comparisons between baseline and summative data, spanning a calendar year, indicate that teachers' beliefs and instructional practices changed to incorporate, to various degrees, a theoretical framework that embodies the belief that teachers' instructional decision making is based on consideration of students' thinking.*

Theoretical Rationale

The importance of considering beliefs when discussing change in decision making cannot be overestimated. However, a careful analysis of teachers' beliefs should not occur in isolation, but in relation to the observed instructional practices (Thompson, 1992). Even though some researchers acknowledge that "beliefs are far more influential than knowledge in determining how individuals organize and define tasks and problems" (Pajares, 1992, p. 311) within their classrooms, other researchers recognize that content knowledge is also related to the decision-making process (Fennema & Loef, 1992). Thus, it is necessary to consider the interaction between teachers' beliefs and knowledge as they are related to practice.

Findings from studies in mathematics involving beliefs and knowledge related to teaching and learning have implications for instructional practice (Fennema & Carpenter 1988; Cobb, Yackel, & Wood, 1988: Fuson, 1990; Post, Harel, Behr, & Lesh, 1988). These studies incorporate a well-defined theoretical framework which can be used to describe teacher change in relation to beliefs, knowledge, and instructional practice.

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Research that has reported a high degree of success in changing teachers' beliefs and instructional practices has been conducted at the University of Wisconsin (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). The Cognitively Guided Instruction (CGI) project is based on the assumptions that children's thinking can be useful to teachers and "that just as children interpret and make sense of new knowledge in light of their existing knowledge and beliefs, so do teachers" (Peterson, Fennema, & Carpenter, 1991, p.108). The CGI research however, focused primarily on first-grade teachers. Little data exists describing teachers' instructional decision making and teaching practices prior to having access to research on children's thinking.

This study intended to extend studies involving the consideration of students' thinking about mathematics in relation to instructional decision making in primarily two ways. First, teacher participants were not limited to one particular grade level but taught grades K-4 and second, data were collected on teachers' teaching methods prior to an intervention. It is not clear that any other study has considered teachers' beliefs, knowledge, and practices K-4 in mathematics and in relation to students' thinking both prior to and after an intervention.

Method

Eleven teachers from a rural K-4 elementary school in central Illinois participated in a year-long inservice project along with their principal. The objectives of the project were to:

1. Provide K-4 teachers with information from research on how children learn mathematics.
2. Assist experienced teachers in creating learning environments that encourage teaching mathematics with understanding by focusing on problem solving and children's thinking.

A belief survey was administered to the teachers both prior to and following the inservice. This survey, consisting of four constructs, was adapted from a measurement instrument of the Cognitively Guided Instruction Project (Fennema, Carpenter & Peterson, 1989). Each statement of the 48-item instrument could be categorized under a construct involving either the teaching or learning of mathematics. Each construct can be scored and placed on a Likert Scale ranging from less cognitively based, with the student viewed as receiving knowledge and the teacher viewed as providing knowledge, to a more cognitively-based perspective, reflecting the belief that children construct their own knowledge and the teachers' role is to facilitate the construction of knowledge. A pre- and post-inservice mathematics lesson was videotaped by the principal. Each teacher responded to a series of questions in relation to why they made the decisions they did with regard to planning for the video. In addition, they were asked to view their videotapes and discuss their decision making during instruction by responding to written probes.

The inservice sessions emphasized recent research findings concerning students' thinking in relation to addition and subtraction of whole numbers (Carpenter & Moser, 1984); multiplication and division of whole numbers (Kouba, 1989); place value (Fennema & Carpenter, 1988); and fractions (Mack, 1990). A one-week summer inservice provided the teachers with an opportunity to share their ideas on how to use the information from research to plan for mathematics instruction.

To understand how the inservices might affect change in instructional practices, belief surveys were analyzed and classroom discourse was examined in terms of the quantitative measurement, time. Teacher-talk time was compared to teacher-student and student-student interaction time. In order to obtain a clearer picture of classroom practices, case studies were also conducted.

Findings

In regard to the belief survey, a two-tailed t-test on all the constructs and the means for all 11 subjects showed significant change in reported beliefs at the .05 level. We believe this change of reported beliefs is not in itself relevant unless it is discussed in relation to the instructional decisions that the teachers were making both before and after the inservice project.

Pre-inservice lessons revealed teacher-talk time ranged from 27% to 99% of total lesson time. The average clustered at 92%. Post-inservice lessons averaged 22% teacher talk-time with a range from 6% to 90%. Not only did the type of classroom discourse change, but also the nature of that discourse. Pre-inservice lessons were characterized by content and knowledge level questions. For example, one first-grade teacher asked, "How many counters are in this group?" during a lesson on part/whole relationships. "Where do I put the comma?" was asked during a lesson demonstrating multidigit multiplication in one fourth-grade classroom. In these pre-intervention classrooms, students were expected to respond to teacher questions, but there was little or no opportunity for discussion or interaction. In contrast, the post-inservice lessons encouraged higher-level questions that required students to explain their thinking. During a third-grade lesson, typical questions posed included: "How did you think about it?"; "Three times? If not, why?"; "Is there a pattern?" and "Why is multiplication like addition?"

All of the pre-inservice lessons focused on activities or procedures and showed little or no evidence that children's thinking was used in making instructional decisions. For example, a fourth-grade teacher talked through a procedure for multiplication and directed the students to "Read rule number 1. Draw a grid like that on the page and then read rule number 2"; or a second-grade teacher who practiced trading ones for tens with straws and as children manipulated the straws, the teacher encouraged students to "tell her how to do it." By contrast, ten of the eleven teachers in the post-inservice lessons were categorized as "cognitively-based" in their instructional decision-making practices. Student thinking and discussion of multiple solutions and strategies were the focus of their lessons. One teacher wrote, "I knew they already had a wide variety of strategies. . . to solve problems" as she prepared a variety of problems to be solved in context, and later lead to the exclamation, "We found three ways to do this problem!" Numbers included in word problems were challenging but also reflected the teacher's awareness of students' abilities: " I tried to give some higher level as well as lower level thinking skills to meet the needs of as many students as possible," reported a first-grade teacher. This teacher encouraged students to share their solutions with the group as a whole, with at least three children sharing different solutions for each problem. Mathematical concepts were being taught via problem solving in nine of the eleven post-inservice lessons. For example, measurement concepts were incorporated into a problem posed to a fourth grade where small groups explored "How many will fit on the desk?" with a variety of materials used to cover the desks. In a third grade class, an investigation of factors of 100 developed from an initial problem involving repeated subtraction. These results illustrate a shift in emphasis and focus regarding both beliefs and instructional practices.

Case Studies

There is not a prescription for teaching in a CGI classroom (Fennema, Carpenter, & Peterson, 1989). Individual teaching styles evolve so that teachers feel comfortable with what they're doing and what their students are learning. Two first-grade teachers, both with 3 years of teaching experience, were selected for case studies based on similar post-intervention belief survey scores and the fact that each taught the same grade. We felt that comparing the practices of these teachers might provide insights into similarities and differences as they implemented and adapted research information to their classroom situation. After the inservices, both teachers relied less on textbooks and more on their knowledge of children's thinking in making decisions concerning mathematical content, selection of problem types, and numbers used in word problems. Both teachers encouraged and listened to students' solution strategies. This was a contrast from their pre-inservice lessons when both teachers monopolized over 90% of the talk time. Both teachers were careful that symbolic notation matched the students' thinking. The focus of instruction changed from tasks that were textbook-driven to student-driven problem solving. The classroom discourse changed from teacher-centered "telling" to teacher-student and student-student interaction that promoted sharing and construction of knowledge.

Discussion

This project provides an example of how teacher change is effected when K-4 teachers are provided access to a well-organized body of research and knowledge, are provided with opportunities to "construct" their ideas of what is meant by effective instruction, and are given administrative and peer support. Without long-term support, changes in beliefs and practices are difficult to effect. In this project, the principal attended all in-service sessions, encouraged teachers to participate in peer-coaching activities, provided feedback to teachers on a regular basis, and communicated with parents and the school board regarding changes in instructional practices. This type of change can empower teachers to make instructional decisions that have the potential to alter curriculum, students' mathematical disposition, school mathematics, and ultimately affect student learning outcomes positively.

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CONNECTING ORIENTATION TOWARDS AUTHORITY TO FIRST-YEAR TEACHERS' THINKING ABOUT TEACHING*

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Three secondary mathematics teachers were observed and interviewed over the course of their first year of teaching. The teachers had participated in a study of the effect of and change in teachers' belief during the final year of their preservice teacher education program. Their experience during their first-year of teaching was analyzed in terms of their orientation to authority, that is, how they come to accept ideas as true or right. The effect of this orientation on their teaching and interactions with students and peers and developmental changes that may have taken place were considered. The paper argues that orientation to authority provides insight into the process of teacher development.

Constructivist epistemology and current reform movements point us to the idea that students learn through interacting with their environment and developing viable constructs to help them organize their experience (von Glasersfeld, 1987). If we consider learning an adaptive activity, then we should think of learning to teach as adaptive as well. Research has indicated that teachers' development is strongly affected by the beliefs and belief structures developed prior to their teacher education (Thompson, 1992). Most would agree that two teachers who go through the same teacher education program and have similar teaching situations may be very different teachers with different values. A problem for teacher education reformers, then is to develop a way of conceptualizing dimensions belief structure differences and to consider how these differences may inform teacher education.

A way of thinking about this developmental process is in terms of what basis a person comes to accept an idea as true or right, that is, the person's orientation to authority (Cooney, 1993). For example, if a teacher values a certain practice in teaching, their willingness take the risks needed to make the practice work in the classroom may be related to where the teacher sees the authority for that practice to rest—internally, through experience and evaluation, or externally, having been told the practice would work. Several theoretical perspectives on orientation to authority have been developed and used by researchers (see Shealy, 1994). The schemes of Perry (1970) and Belenky, Clinchy, Goldberger, and Tarule (1986) are noteworthy because of the developmental stance taken. In the early, dualistic stages of these developmental perspectives, the individual sees all truth and knowledge coming from outside of self and believes all questions have a single right answer. The individual then begins to hear competing voices and deals with them by explaining the differences away or, eventually, accepting that all opinions are right and begins to find his or her own voice (multiplism). Next, the individual begins a struggle to deal with these voices and evaluate and integrate them with his or her own, thus, moving into relativism. In the final stages, the individual becomes more comfortable with this integrating process and is willing to take the risks involved with commitment. Belenky et al.'s scheme was informed by that of Perry, but improves upon it by emphasizing and elaborating on the process of integrating other voices in the later stages, that is Perry's scheme seems to have a sense of moving toward self-sufficiency rather than connection.

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In a previous study (Shealy, Arvold, Zheng, & Cooney, 1994), we investigated orientation to authority with preservice secondary mathematics teachers. In considering studies that show the incongruity between belief and practice, particularly newly acquired beliefs or the rejection of orientations in the context of first-year teaching (Thompson, 1992), it seems important to go beyond the preservice experience of the teachers and consider the thinking of teachers during their first-year of teaching. This study addresses the question of how understanding teachers' positions with respect to authority brought into their teaching help us understand their thinking about and planning for instruction and the perseverance and enaction of beliefs developed during their teacher education program. Conversely, this study considers how the context of first-year teaching affects the teachers' orientation to authority.

Methods

The three participants in this study were a part of the preservice teacher education study (Shealy et al., 1994). In this study, we were concerned with conceptualizing the teacher development process in terms of changes in inferred beliefs and belief structures connected to mathematics and teaching and learning mathematics. Interviews, questionnaires, and observations of teaching and interaction with peers were used to address how these beliefs were connected to their experience before their teacher education, how the beliefs were organized, and how we could characterize movement in these inferred belief systems. As the studies progressed, the notion of authority moved to the foreground in our thinking. The present study follows three of the teachers into their first year of teaching.

The data for this study consists of field notes, interviews, and various written artifacts (e.g., tests, student assignments, reports to administrators) from three periods, one each in the fall, winter, and spring. These data collection periods were selected to encompass crucial times in the participants' development as teachers. The fall period (October, 1993) was scheduled early enough to see the teachers struggling to adjust to their new environments. The winter data collection (January, 1994) occurred at the end of their first semester of teaching. The spring visit (April, 1994) was scheduled near the end of the school year. During each period, the teacher's classes and interaction with peers and students were observed for three to five days. Each teacher participated in two or three one-hour-long interviews during each period. The interviews were primarily open-ended, but informed by interview guides. One participant, Nancy, had a difficult experience adjusting to teaching and asked to discontinue her participation in the study, but later agreed to participate in two interviews, one each during the fall and spring.

Case Studies

All three participants taught in rapidly growing suburban school districts in a major metropolitan area. Todd and Greg taught in high schools while Nancy taught in a middle school. All three had grown up in or near the districts where they taught, attended the same nearby major university, and shared their teacher education courses. Todd and Greg taught geometry, algebra or prealgebra, and a remedial course. Nancy taught algebra and prealgebra to eighth graders.

Nancy

One noticeable theme in Nancy's thinking is her desire to be like others—her mother and aunt who are mathematics teachers, a favorite high school teacher, her instructors in mathematics education, and her peers in her teacher education classes. Nancy recognized multiple perspectives on mathematics and teaching mathematics, but was looking for, in a sense, the "right way" to teach. During her teacher education program, Nancy saw herself as a learner. She valued her instructors and the consensus of her peers. She was surprised, however, when one peer expressed views that seemed radically different from the rest of the group. She rationalized the difference as his "saying the same thing" in a different way or maybe with a different style.

When Nancy began teaching she expressed the desire that her students like her and thus want to do well for her. Quickly, Nancy had difficulty preparing for class and controlling her students. She became extremely distraught and considered quitting teaching. Although initially enthusiastic about this study, she struggled with whether to continue to participate, afraid that she "wouldn't be much help." At this time her mother offered to come and observe her classes as well, but Nancy did not want her in the classroom either. She said, "I felt like I was just awful and it was just horrible and I would be so embarrassed if anybody saw me do this right now." Her reluctance to be observed related to a feeling of letting down those she respects.

When Nancy experienced trouble initially, she went to her administrators and peers. Whatever they suggested, she tried. She struggled with feelings that something was wrong with her—that she had not learned well enough. Later in the year, Nancy went from writing a script for her class and following it word-for-word to being "mean Miss [Nancy]" and establishing a routine and some classroom discipline. She said, "I would take the problems home with me at night and I would have nightmares. . . . Finally, I got the point [where I said], 'Why am I letting these little eighth graders take over my life?' "

Nancy began her teacher education program with an essentially dualistic view of teaching. She saw herself as a student ready to be filled. Like Belenky et al.'s (1986) received knower she was ready to learn and be made into a teacher. Through interaction with peers she came to accept multiplistic ideas uneasily. She was amazed at people who could be creative and come up with good ideas for the classroom. When she started teaching, she was embarrassed at her failure. As peers and administrators could not help, Nancy was forced to either quit or begin to find her own voice and style in the classroom. Nancy still struggles with finding her own voice, but is no longer contemplating leaving teaching.

Todd

During Todd's teacher education experience, he talked about trying to understand how the students are thinking, he appreciated different ways of approaching problems and concepts in mathematics, and emphasized students as active learners. Todd had no trouble seeing that people had different views on teaching and mathematics and hoped, before he started teaching, to help students express their own ideas. He had accepted the value of open-ended investigation and problem-solving he experienced in his teacher education program.

Once Todd started teaching, although, he said, "everybody has their own ideas" and that he hoped to share his ideas but didn't want to "step in toes," he often criticized the other teachers in his school as "afraid of change." He talks about getting advice from other teachers, but either criticizes the advice or characterizes it as something he already knew. In a sense, Todd sees himself as belonging to the "in" group in educational reform and sees himself as an agent of change, while his peer teachers need to "catch up."

In the classroom, early in the year, Todd tried open-ended investigations, problem-solving activities, and activities designed to allow the students to take authority for the class. Todd concluded that these activities did not work. He found some of the activities to be trivial, some were frustrating to the students, and at other times, he lost control over the classroom. His conclusion was that the activities stressed in his teacher education program were for advanced students and that students get frustrated if you do not quickly "give them the next step" in solving a problem. Thus, Todd, who had extensive experience tutoring at a learning center during his college years, began to set aside a significant portion of class time for students to work individually with his help. At the end of the year, Todd said he planned to teach a few more years and then begin a career of developing educational software. This career shift would allow him to pursue his ideas without the constraints of the classroom.

Todd hears competing voices and through his teacher education courses and a summer institute has found his own voice. In the more multiplistic stages of development, sorting out the various voices can be threatening. Finding one's own voice provides a place to stand, however tenuous it may seem. Like the subjective knower (Belenky et al., 1986), Todd professes "to be open to anything, [but be], in fact, stubbornly immune to other people's ideas" (p. 98). Then when the activities Todd enjoyed in his teacher education courses do not work for his classes, he attributes the failure to the activities or the students and quickly falls into the pattern of instruction he is used to without seeing a contradiction to his expressed values. Also indicative of Todd's development, Perry (1970) says, someone in multiplism may use the multiplism as an escape form responsibility to pursue their own ideas. This escape is the direction Todd seems to be going.

Greg

Early in his teacher education experience, Greg expressed ideas that appeared dualistic (Perry, 1970). For example, he saw his calling to teach as a mission he had to follow and he was adamantly opposed to the use of calculators in the classroom. Further observation and interviews seemed to indicate Greg was on or close to Perry's (1970) commitment levels. He says, for example, "I do have set beliefs and I know that that's not always the best way to handle things and I'm flexible enough to know that my way of doing things doesn't work all the time." Greg hears and values others voices but takes the risks of commitment.

In Greg's planning for teaching he tries to find activities for the students to be involved in investigation. In many of his classes, a significant portion of the class time is set aside for investigation. He is excited about the textbooks he is using, but is willing to question the presentation and modify or even skip a section if he feels it is appropriate. After participating in evaluating another textbook, he said it was important to hear what others say and incorporate new views, but "you have to be the final judge for what you do in the classroom."

As a teacher, Greg says he has to "provide an environment where [the students] can learn." A part of this environment is "to let and help [the student] express [him or herself]." In order to establish this environment, Greg realizes he has to take risks. He has seen lessons not go well and has lost control of classes. Greg is not, however, discouraged by these instances. He is willing to take pains to act on his beliefs. For example, during Greg's teacher education program he became convinced of the usefulness of various tools. During his first year of teaching, he suspected that a MIRA would be useful for his geometry class. He then took the time to learn how to use the MIRA and developed lessons for his class to use.

Greg also strongly values others ideas. This value has led him a have interaction with many teachers and good working relationships with his peers. He does not always agree with his peers, but believes it is important to listen to others. This valuing of others' voices, dialectical thought, and willingness to take risks of commitment are all characteristics of Perry's commitment stages and Belenky et al.'s constructed knower. Belenky et al. say that constructed knowers, "aspire to work that contributes to empowerment and improvement in the quality of life for others" (p. 152). Greg's lessons do not all go well, he has classroom discipline problems, but none of these problems stand out in his mind.

Conclusion

Orientation to authority as described by the frameworks of Perry (1970) and Belenky et al. (1986), provides ways to think about how teachers respond to and use their teacher education experiences, especially as they move into their first year of teacher. Greg's ability to contextualize mathematics and teaching and exercise internal authority contributes to his confidence and flexibility as a teacher and his good working relationship with peers. As a subjective knower, Todd's clinging to his own voice, tends to isolate him and causes him to quickly reject practices that are not initially successful in the classroom. Nancy's dependence on external authority seems to limit her ability to deal with situations as they arise in the classroom, causing her to blame herself when things do not go well.. The question for teacher education, then, is what contexts do each of these teachers need due to their orientation to authority and to promote growth in this area?

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A FIRST YEAR SECONDARY MATHEMATICS TEACHER'S ORIENTATION TOWARD AUTHORITY: IMPLICATIONS FOR PROFESSIONAL DEVELOPMENT

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During a year long study on the evolution of teachers' beliefs during a teacher education program, we became intrigued with Henry. He consistently manifested dualistic (Perry, 1970) beliefs about mathematics and the teaching of mathematics. (See Shealy, Arvold, Zheng, and Cooney, 1993.) Because of our high fidelity (Noddings, 1986) relationship with Henry and our interest in possible modifications of his beliefs as a result of his first year teaching experiences, we continued our investigation of him. Our study included October, January, and April structured telephone interviews and a full day of personal interviews, observation, and discussion in June.

Throughout his first year of teaching, Henry's orientation to authority continued to manifest itself through his relationships with people and ideas. His perception of the world as an unfriendly place, a basis for dogmatism (Rokeach, 1960) permeated his discussions of mathematics and teaching. Having been assigned five classes of lower ability students, Henry felt that he was denied the opportunity to teach mathematics. Although he was a strong proponent of students practicing basic skills and believed his students needed this practice, he steadfastly followed the textbook, which in his view dealt with real life skills instead of mathematics. Matters dealing with authority in the classroom continued to be a concern with Henry. When he encountered difficulty dealing with classroom situations he placed the blame on a deficient teacher education program. Early in the year his administrators began responding to his need for assistance by intervening in disciplinary matters and devoting a full time aide to Henry's classes. It is no surprise that Henry described his first year teaching as much easier than he expected.

Interestingly, he did not accept a position in a summer institute which would have insured him of a higher level mathematics teaching assignment. Yet at year's end, knowing full well that he would have three preparations and no teacher's aide, he accepted a position in a school known for promoting tenets of reform. Although Henry's dualistic orientation intensified during his first year of teaching, he also began to reflect positively upon many of his university experiences. We now question how this initial foray into reflection might impact his professional development.

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EMERGING AND EVOLVING THEMES: CASE STUDIES OF THREE PRESERVICE SECONDARY MATHEMATICS TEACHERS' BELIEFS

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Case studies were carried out to identify and compare emerging and evolving themes in the models for teaching and learning mathematics held by three preservice teachers during their first six months in a two-year development program. The researcher was a participant observer.

The research framework was based on an inclusive perspective toward teachers' beliefs, identifying characteristics of beliefs described by Green (1971) and Nespor (1987). Ernest's (1989) model was used to specify the domain of beliefs under investigation.

Data were collected and analyzed in three phases: (I) As they began program involvement, data from all 11 program participants were used to select 3 subjects. The three subjects then completed a two-part repertory grid interview to identify emerging themes in their models for teaching and learning mathematics. (II) For six months, data were collected to monitor subjects' themes. Data included program artifacts, interview transcripts, and observations of on-campus and off-campus activities. (III) At the end of six months, data were collected to identify subjects' evolving themes. Descriptions were based on factors substantiated during repeat repertory grid interviews conducted at the close of the study.

The subjects shared six emerging themes: classroom relationships, management, guided-discovery activities, assessment, communication, and motivation. Emerging themes differed in one subject's conception of student learning and another subject's recognition of being in transition. The emerging themes tended to emphasize structure and affect. As bases for their preferences and intentions, subjects referred to course work, field experiences, and prior experiences as students. All subjects spoke of school mathematics courses wherein instructors had presented fixed approaches to producing solutions.

Subjects shared four evolving themes: student learning, students' active involvement, management, and motivation. Evolving themes showed a shift in emphases. There was less commonality across themes, and the evolving themes that shared elements with the emerging themes — management, learning activities, motivation — were identified from a new perspective. Preferences, intentions, and concerns were provoked by experiences with school students.

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WARNING—HARD HAT AREA: FALLING BELIEFS

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Beliefs about mathematics, mathematics teaching, and mathematics learning of approximately 160 preservice elementary teachers at a midwestern public university were investigated before and after a two-semester sequence of required mathematics content courses. The five-point, Likert-type beliefs inventory that was administered included the following eight belief scales:

- I can solve time-consuming mathematics problems. (Difficult)
- There are word problems that cannot be solved with simple, step-by-step procedures. (Steps)
- Understanding concepts is important in mathematics. (Understand)
- Effort can increase mathematical ability. (Effort)
- Mathematics, in general, is useful. (Useful)
- The mathematics I'm studying this semester is useful to me. (Usecrs)
- Mathematics is enjoyable to me. (Enjoy)
- I am confident that I can teach elementary school mathematics. (Teachcnf)

Changes in beliefs occurred over the two-semester period of the study, supporting the view of Thompson (1992) that "belief systems are dynamic, permeable mental structures, susceptible to change in light of experience" (p. 140). Significant changes ($p<.001$ or $p<.05$) were found in five of the belief scales: Useful, Usecrs, Enjoy, Steps, and Understand. Although mean scale scores indicated that students' beliefs were mostly positive, four of the five significant changes that occurred were in a direction indicating deteriorating beliefs. The only significant change that occurred in the direction of enhanced beliefs was on the Useful scale.

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Teacher Change

RESEARCH PAPERS

Transforming Mathematics Teaching In Grades K-8 189
Linda Ruiz Davenport

Affective Issues In Developing Math. Teaching Practice 196
Lynn Goldsmith & L. Davenport

Case Studies On Empowering Secondary Mathematics Teachers In Computer Environments 203
M. Kathleen Heid, Rose Mary Zbiek, and Glendon Blume

Perceived Deficits In Middle Grades Mathematics Teaching 210
Kathleen Heid & Susan Feeley

How Middle School Teachers Adjust To Change 217
Diana Lambdin & Ronald Preston

The Needs of Second Career, Secondary Math. Teachers 224
Sheryl A. Maxwell

Development Of Classroom Social Norms 231
Betsy McNeal and Martin A. Simon

Teaching In An Era Of Reform: In Elementary Classrooms 238
Paola Sztajn and Frank K. Lester

Changing Practice: Perry's Scheme 245
Melvin Wilson & M. Goldenberg

ORAL REPORTS

Teacher Paradigm: Form And Transition 252
Sharon B. Walen

POSTERS

Extending K-8 Teacher Development In Mathematics To The Secondary Level: A Model For District Systemic Reform 253
Emily Dann, Roberta Schorr, and Alice Alston

Mississippi Mathematics Improvement in Secondary Schools Project (MISS) 254
Barbara Dougherty & Larry Tyler

A Reflective Cycle: The Evolution Of A Model Of Teacher Change 255
Thomas G. Edwards

Teacher Inquiry Groups: Collaborative Explorations Of Changing Practice 256
Jim Hammerman

Negotiating A Technological Shift: Teacher Perception Of The Implementation Of Graphing Calculators In The Classroom 257
Marcia L. Tharp, Robin L. Brown, and James A. FitzSimmons

TRANSFORMING MATHEMATICS TEACHING IN GRADES K-8: THE ROLE OF MATERIAL RESOURCES IN SUPPORTING TEACHER CHANGE

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Abstract

This study examined the role of material resources in supporting teachers attempting to transform their mathematics teaching practice. Teachers in a larger funded project with regular access to a wide range of resources were asked to identify and discuss resources of significance to them. While teachers identified resources across a range of categories, most conveyed detailed information about other teachers' classrooms or contained numerous examples of student work. These findings suggest that resources that provide concrete images of teachers and students exploring this new way of doing mathematics can be of great value to teachers.

A vast number of material resources are now available to teachers who seek to create in their classroom the vision contained in the recent *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) and the *Professional Standards for Teaching Mathematics* (NCTM, 1991). Many of these resources come from NCTM itself, such as the "Implementing" columns in *The Arithmetic Teacher* and *The Mathematics Teacher*, the recent *Addenda Series* , and other publications addressing issues of relevance to reforming mathematics teaching practice. Many other materials are also available through other publishers and projects.

Although we still have a great deal to learn about the teacher change process, it is clear that the kind of change called for in the NCTM *Standards* involves more than assimilating new teaching techniques in an existing system of ideas (Cohen et al., 1993). Rather, it requires the critical examination of long-standing beliefs about what it really means to teach mathematics, an epistemological shift involving the creation of a new vision of students as constructors of their own knowledge, and the reinvention of pedagogical strategies and approaches consistent with that vision (Schifter and Fosnot, 1993; Simon and Schifter, 1991). Unfortunately, we know little about the role of material resources in prompting and supporting this kind of teacher change.

This small exploratory study was designed to examine the role of material resources in helping teachers make these fundamental changes in the way they think about mathematics, mathematics learning, and mathematics teaching. It was set in the context of an already-funded teacher change project explicitly designed to provide teachers with opportunities to explore and discuss material resources in after-school inquiry group meetings and over an electronic network. Given the epistemological shift required in the teacher change process, a central question was whether or not teachers would choose resources that challenged their beliefs and practices and could lead to the development of new ways of thinking about their mathematics teaching practice.

Methodology

A total of 34 teachers from grades K- 8 in four school districts were involved in this study. These teachers were all participants in the Teachers' Resources Network (TRN), a project funded by the Dewitt Wallace-Readers' Digest Fund. The resource collection which is part of this project contains approximately 600 items organized into the following categories: mathematics, kids' understanding, assessment, diversity, teacher practice, teacher change, and teaching and learning.[1] Teachers explore these resources and their implications for mathematics teaching in biweekly inquiry group meetings, with additional opportunities to discuss issues over an electronic network. All teachers keep portfolios which include reactions to resources, reflections on inquiry group discussions, and journals on their classroom practice.

Although documentation of teachers use of and reactions to material resources is being collected using a range of strategies, this paper focuses on teachers' writing from their summer conference. On the final morning of this conference they were asked to review the contents of their portfolios, list the resources they had reviewed, select one that was most significant for them, and write at least a page about why it was significant.

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[1] These were chosen in an effort to provide some initial organization for the resources, and the boundaries between these are actually difficult to draw. For example, the category "teacher practice" was intended to contain resources on pragmatic aspects of mathematics teaching practice, "teacher change" was intended to contain resources on the change process itself, and "teaching and learning" was intended to be more theoretical in its focus.

Results

The number of resources each teacher explored ranged from 0 to 10. Six teachers reviewed no resources[2]; 15 teachers reviewed one, 2, or 3 resources; 6 reviewed 4 or 5 resources; and 7 reviewed between 6 and 10 resources. The 106 resources that were reviewed fall into the following categories:

Curriculum	25	23.6%
Kids' understanding	21	19.8%
Assessment	21	19.8%
Teacher change	16	15.1%
Teaching practice	16	15.1%
Diversity	6	5.7%
Teaching and learning	1	0.1%
Mathematics	0	0.0%

Only 22 of the 34 teachers identified a resource as being of particular significance to them. These significant resources were spread as follows:

Curriculum	7	31.8%
Kids' understanding	5	22.7%
Teacher change	5	22.7%
Assessment	2	9.1%
Teaching practice	2	9.1%
Diversity	1	4.5%
Teaching and learning	0	0.0%
Mathematics	0	0.0%

Since these categories may mask similarities among resources, features of the kinds of resources identified as significant within each are described below.

Curriculum: The resources identified here include a set of middle school mathematics activities (Bennett et al., 1986) identified by two teachers, a book suggesting strategies for integrating literature and mathematics (Whitin and Wilde, 1992) also identified by two teachers, a "replacement unit" for teaching place value at the second grade level (Burns, 1994), a book chapter on place value activities for young children (Thompson, 1990), and a set of puzzle problems for exploring area and perimeter (Dominguez and Laycock, 1986). What is notable about many of these curriculum resources is that all but the last contain discussions of how students are likely to think about or respond to the suggested explorations. In fact, the literature and mathematics book and the replacement unit contain stories of teachers using the activities and include samples of classroom dialogue as well as student work.

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[2] Five of these teachers joined the project late in the spring and chose to spend their time learning to use the telecommunications network. One teacher, because of personal problems, did not check out any resources although he attended all of the inquiry group meetings.

Kids' understanding. These resources include a book about the mathematical thinking of young children as it emerges through student interviews (Labinowicz, 1986) and an article describing how students invent algorithms for basic operations (Kamii et al., 1993) which also provides numerous examples of these algorithms. Each of these resources was also identified by two teachers. The other resource in this category is a videotape showing two teachers working with students as they invent algorithms for 2-digit multiplication (Kamii, 1990).

Teacher change. This includes an article about a teacher who thought she had "reformed" her mathematics teaching but whose practice still resembled the traditional paradigm in significant ways (Cohen, 1991), an article about the role of teacher reflection in the teacher change process (Feldt, 1993), and three papers by teachers transforming their mathematics classroom practice (Brown, in press; O'Brien, in press; Schott, 1992).

Assessment. Here, the significant resources consisted of a practical guide to alternative assessment that includes a great number of examples of student work (Stenmark, 1991), and an article in which a classroom teacher writes about her efforts to use portfolio assessment, and which also contains examples of student work (Lambdin and Walker, 1994).

Teacher practice and diversity. The last remaining resources identified as significant include a discussion of the challenges facing educators today, a review of literature on cooperative learning, and an article on how teachers' attitudes towards mathematics are reflected in their teaching practice.

Teachers explanations of why these resources were significant, although often brief and not always not to the point,[3] include the following:

> This resource was significant for me because it gave me a very specific structure and outline for my entry into drastically changing my way of covering the math curriculum. This book was a hand to hold on my journey into giving up mandated curriculum . . .
>
> (This resource) provided insight into the wealth of knowledge children actually have if allowed to show it. We, as teachers, so often stifle that creativity and don't allow children to be "smart".
>
> I really liked this article because the author approached algebra much like I do. . . She wanted students to explore concepts, discuss patterns, and form generalizations. She did not just give formulas and have students plug in numbers. She also explored and understood the resistance from students and parents. . .
>
> This article . . . encourages a teacher to interact with students and be more constructive in their methods of teaching. I would like to be a creative teacher and not just a drill master. . .
>
> This resource really affected me was because I feel that my attitudes toward math affect the way that I present math in my classroom. . . This article made me think about how I pose questions. . . This resource really got me thinking. . .

In most cases, teachers wrote that they were able to see some aspects of their own practice, or some aspects of what they hoped might become their practice, in these resources.

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[3] Just as teachers find that their students are initially unaccustomed to writing about their mathematical thinking, they same might be said of teachers and their thinking about their mathematics teaching practice. This was the first writing activity of this sort for these teachers in the project, and it was clearly uncomfortable and difficult for them.

Conclusion

Generally, teachers explored resources in the areas of curriculum, kids' understanding, assessment, the teacher change process, and teaching practice. There was less interest in resources which specifically addressed diversity issues, and little interest in resources of a more theoretical nature (e.g., constructivism) or resources about mathematical topics (e.g., the meaning of a variable). Of the resources that teachers reported as significant, 18 of the 22 resources either contained stories from classrooms that conveyed images of how teachers and students work together or included numerous examples of students' mathematical thinking. Teachers reported that these resources gave them ideas for the kind of mathematics teaching and learning that might take place in their own classrooms. In at least one case, a teacher reported that a resource helped them think about their own practice in a new way.

Since this study was undertaken in the early stages of project, a number of questions remain to be examined. In what ways will the resources teachers select and identify as significant change over time, as their teaching practice changes and develops? In what ways will teachers' abilities to become more articulate about the significance of these resources allow us to better understand the role they play in prompting and supporting teacher change? Finally, it is important to note that the findings of this study do not suggest that the exploration of resources in and of itself is sufficient to prompt and support teacher change, but that combined with a collegial group setting and opportunities to explore mathematics, they may help provide windows into what good mathematics practice looks like, sounds like, and feels like.

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AFFECTIVE ISSUES IN DEVELOPING MATHEMATICS TEACHING PRACTICE

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Abstract

This papers identifies a set of research questions for exploring the role of affect in teacher development. These questions emerged from an examination of data from an extended teacher development project based on constructivist principles. Although these data were not collected with the intent of specifically examining the role of affect in teacher development, they suggest issues that might fruitfully be examined to better understand what teachers feel as well as what they think as they attempt to transform their mathematics teaching practice.

What kinds of emotions do teachers experience as they work to change well-established teaching practice? In calling for substantial changes in the kinds of mathematics students learn, the current reform movement in mathematics education also calls for substantial changes in the ways mathematics is taught. Teachers are not being asked simply to create new lessons for new curriculum topics, but to teach in new, qualitatively different ways. Teachers often find that this process challenges their fundamental beliefs about learning and teaching, the nature of mathematics, and the nature of mathematical understanding.

Making such changes is no easy task, even for teachers who strongly wish to do so. Teachers often confront doubts about their professional effectiveness as they reflect on their own classroom practice. They must be willing to take risks in their classrooms, setting aside familiar and well-established classroom structures and activities in order to explore new ways of thinking and learning about mathematical ideas. Furthermore, teachers rarely have the luxury of embarking on such a reconstruction of practice in an atmosphere that supports inquiry, and reflection: they undertake this kind of examination and experimentation within the context of ongoing classroom responsibilities, where the desire to serve their students creates counter pressure for significant change (Cohen et al., 1990). Transforming teaching practice under these circumstances has been likened to redesigning and rebuilding an airplane while in flight.

These circumstances are almost guaranteed to generate strong feelings. Observations of teachers, as well as their own anecdotal self-reports, suggest that the process of change invokes a variety of feelings: it is variously exhilarating, intimidating, frustrating, invigorating, discouraging, and immensely satisfying. But while researchers and teacher educators are aware that changing practice stimulates strong emotions, there has been little systematic study of this issue. In fact, relatively little attention has been paid to the more general question of how cognitive and affective states interact in naturally occurring settings over extended periods of time, despite recent acknowledgment that mathematics educators must attend more explicitly to the role of affect in learning (Kouba, 1993; McLeod & Adams, 1989; Sowder, 1989).

This paper reports on an effort to study affective aspects of changing mathematics teaching practice. Unlike recent work on local affect which examines how the emotions that students experience in the course of working on mathematics problems might directly influence these problem-solving occasions (Goldin, 1988; McLeod, 1988), this work explores the ebb and flow of emotion over an extended period of time--months instead of minutes--and focuses on the interaction between thinking and feeling within the naturalistic context of teachers' professional practice.

Methodology

Exploration of the emotions of teacher change must be embedded in investigations of the change process itself. Therefore, we sought to collect information from teachers who were participating in an extended professional development project that was designed around constructivist principles of learning and teaching. Data for this paper came from artifacts produced by middle school teachers participating in a national five-year project designed to "foster and support the implementation of innovative, high-quality mathematics instructional programs" (Silver, 1993). These artifacts included journals, audiotapes and videotapes of teacher group meetings, and videotaped classroom lessons. The context in which these artifacts were produced is well understood by one of the authors of this paper, who served as a resource partner for the participating teachers. While these data sources were not specifically designed to study affective aspects of teacher development, they can be examined for spontaneously occurring evidence of the emotions of change. They provide a starting point for thinking about emotional issues inherent in the teacher change process and can help direct our observations in future work.

Preliminary Findings

Examination of the data led to the identification of a number of research questions about the role that affect plays in the process of reconstructing mathematics teaching practice. Several of these questions, along with preliminary findings and suggested future directions, are presented below.

(1) What are some of the feelings that seem to be commonly experienced as teachers work on transforming their mathematics teaching practice? Project teachers expressed a variety of feelings about their work, ranging from feelings of excitement and anticipation to uneasiness, loneliness, or frustration, although the expression of emotion--and perhaps even the *perception* or acknowledgment of feelings--varied with individuals. Some teachers were very willing to share their feelings about this process through journal writing, by discussing their concerns, frustrations, and pleasures with their colleagues in group meetings, and by displaying their excitement and engagement about teaching mathematics in their classrooms. Other teachers were less emotionally reflective, and less inclined to talk about their feelings, even when asked about them by project staff. Yet, while there were differences in the *extent* to which teachers revealed their emotions, the expression of both positive and negative feelings was common to all.

In the current literature, much of the anecdotal information about the emotions of change has focused on negative aspects of the process: feelings of isolation from other, more "mainstream" teachers; anger at being pushed to think hard about their teaching; anxiety about not teaching well; guilt for having taught children in ways that now seem less than optimal (see, for example, Schifter and Fosnot, 1993). Despite the emphasis on the difficulties involved in undertaking such significant change in practice, this cannot be the whole picture. It seems unlikely that teachers would persevere with such demanding work if the process held no moments of joy, excitement, curiosity, or sense of new understanding and intellectual power. While it is important to understand the ways in which negative emotions may interfere with further growth, it is equally important to understand how the emotional rewards of professional development help to sustain teachers as they inquire into their practice.

(2) *Where does affect get expressed and worked out in this process of teacher development?* The artifacts we examined indicate that, for some teachers, journal writing offered an opportunity to describe, explore, and communicate the feelings associated with change. Some teachers used their journals to write about the learning of individual students, to describe events in the classroom, and to note their feelings about their work and their roles as teachers. For these people, the journals provided a source of information about the psychology of their change process, not only with respect to their affective responses to the endeavor but also with regard to the more cognitive aspects of the process. For other teachers, journals provided little insight into their thoughts or feelings. These teachers used the journal more for record keeping than for reflecting, logging in their lesson plans with little detail about what actually happened in the teaching of the lesson or the kinds of issues about their practice that the lesson surfaced.

We also found a second context for the expression of emotions: the monthly teacher meetings facilitated by the project's resource partners. Here, teachers talked about their classrooms, thought together about ways to improve their practice, and shared the pleasures and frustrations of working on their teaching. Within the context of a supportive group of fellow teachers and facilitators, teachers engaged in the intellectual work necessary for building a new form of practice and also expressed the excitement, anxiety, and frustration about the process--that is, they shared the emotions that accompany being outside of familiar territory and involved in the exploration and construction of something new. As with the journal writing, not all teachers used the opportunity to share emotions in this public forum.

A third and more difficult context to capture was the informal context in which teachers expressed feelings about their practice with colleagues and friends. The teachers in this project all worked in the same building and checked in with each other. They thought of themselves as a team, and felt it was important to listen to and learn from each other. They reported that they often talked with each other about how particular lessons had gone or about specific students, and they even had a name for this kind of checking in--they called it "tagging up." These conversations seemed to be important to the teachers, both for the specific information that they provided and for the psychological support that they implied.

These findings suggest places in which researchers might begin to look for evidence of the affective side of teacher change, and also touch upon the kinds of contexts and relationships that teachers find appropriate for expressing these feelings. Our findings also raise methodological issues about the challenges of capturing affect within this range of settings, because many important affective conversations occur spontaneously, within the school setting, as part of the natural flow of the school day and are difficult to document.

(3) *What kind of role might emotions play in the process of teacher*

development? Project teachers experienced successes and failures in their classrooms; at times they felt energized and excited about their teaching and at other times they felt confused and concerned. At present, we know little about how feelings serve as motivators for change, but we suspect that affect plays a significant role as a motivator for teacher development. What, for example, energizes teachers to make changes in their practice? What helps them sustain their commitment to reflection and experimentation with practice when this often means that classes run less smoothly than before? What kinds of experiences and emotions impede the process of change? In what ways are these feelings indicators of teachers' current level of engagement in the process of change? And in what sense might being able to work through a transformation in professional practice have broader effects on teachers' constructions of their sense of self as competent, autonomous, effective people?

Conclusions

We currently know little about the emotions of teacher development, and we need to know more. This work represents a step in the direction of greater understanding. It raises questions about the impact of emotions on the process of reconstructing mathematics practice itself. By acknowledging the range of feelings that are a legitimate part of this process, it also offers support to teachers expressing their feelings in productive ways. We need to pay closer attention to the kinds of feelings teachers experience, and to the conditions that elicit different feelings. We also need to develop ways of investigating the process: where do we look for the emotions of transforming teaching practice; how much do we rely on teachers' self-reports and how much do we interpret from teachers' behavior; how do we account for individual variability in the expression (and possibly the recognition) of different feelings?

We will not have a complete theoretical picture of the process of change, nor will we have powerful practical models for facilitating change, until we take the affective aspects of the process into consideration. Furthermore, the very modes of teaching that are being advocated by the mathematics reform movement are ones that will generate more opportunities for emotional expression in the classroom by intensifying the nature and degree of intellectual engagement. As both teachers and students engage in thinking more deeply, and more publicly, about mathematical ideas, there will be more occasions for encountering and addressing the feelings of confusion, pride, frustration, and excitement that accompany genuine learning.

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CASE STUDIES ON EMPOWERING SECONDARY MATHEMATICS TEACHERS IN COMPUTER-INTENSIVE ENVIRONMENTS

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This paper describes case studies of teachers in their first efforts to implement a computer-intensive secondary algebra course. It explores interaction among knowledge of mathematics and technology, understanding of student understanding, pedagogy, and assessment in the context of implementing Computer-Intensive Algebra. Teachers participated in a four-week Computer-Intensive Mathematics Education institute on using technology to teach algebra, on using interviews to help them better understand student understanding, and on understanding the mathematics underlying technological approaches to secondary school mathematics. Three interviews and a series of observations focused on teachers' use of technology to explore mathematics, their ability to assess student understanding, and the effects of their assessment of student understanding on the decisions they made in the context of teaching.

Currently available technology suggests dramatic changes in the content, the teaching, and the learning associated with school mathematics (Fey, 1984; Heid, Sheets, Matras, and Menasian, 1988). As computing tools (e.g., graphers, spreadsheets, computer algebra systems) become a more natural and expected feature in the environment of secondary mathematics classrooms, teachers will need to use the technology to promote students' ideas and to act as facilitators of students' technological explorations. To date, little research has been done on empowering teachers in computer-intensive environments.

Researchers (Fennema and Franke, 1992) have suggested that some of the important components of teachers' knowledge that impact on their teaching and their students' learning are: knowledge of mathematics (Ball, 1988; Lampert, 1989) and mathematical representations (Hiebert and Wearne, 1986), pedagogical knowledge (Clark and Peterson, 1986; Shulman, 1986), and knowledge of how students come to understand mathematics (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). In computer-intensive environments, additional components of teachers' knowledge that impact on their teaching and their students' learning may include knowledge of the use of technology for the exploration of mathematics and knowledge of the technology itself.

The setting

In order to examine the empowerment of mathematics teachers in computer-intensive environments, the Empowering Mathematics Teachers in Computer-Intensive Environments project implemented a model that included learning the mathematics that underlies computer-intensive mathematics curricula at the secondary level, learning to use interviews to assess and analyze student understanding in technologically rich classrooms, and implementing *Computer-Intensive Algebra* (Fey, Heid, et al., 1991), a computer-intensive beginning algebra curriculum. Teacher-participants completed a 1993 month-long summer course centered on these components and implemented *Computer-Intensive Algebra* in their classrooms the 1993-1994 academic year before returning for a one-week course in summer 1994. The mathematics component engaged subjects in mathematical activities, including explorations of original families of functions as well as original mathematical modelling projects. The assessment component included examining student interviews and conducting their own interviews with students who had completed *Computer-Intensive Algebra* during the previous academic school year.

The subjects

Three of the thirty participants were identified for case studies because their proximity to the researchers permitted classroom observations on a somewhat regular basis. These subjects (one male and two females) had 21 to 22 years of experience teaching mathematics and had varying degrees of previous familiarity with the use of technology in the teaching of secondary mathematics. Each of the schools in which a subject taught drew students from both small towns and rural setting. None had previously taught *Computer-Intensive Algebra* although one had been using *The Geometer's Sketchpad* (Jackiw, 1991) as a demonstration tool in her Geometry classes.

Task-based and student-understanding interviews

The three subjects each participated in several series of interviews during the summers of 1993 and 1994 as well as during the 1993-1994 academic year. At the beginning and at the end of the summer course, the subjects participated in task-based and student-understanding interviews. Task-based interviews assessed subjects' use of technology and understanding of selected mathematical ideas underlying computer-intensive mathematics curricula. The mathematical ideas on which the interviews focused were function, families of functions, mathematical representations, and mathematical modelling. Interviewees had access to computers and calculators throughout these hour-long interviews. Student-understanding interviews addressed the subjects' ability to understand student understanding. Subjects were given a transcript of a portion of an interview with a *Computer-Intensive Algebra* student and asked to discuss the student interviewee's mathematical understanding based on evidence in the transcript.

Classroom observations and interviews

At four times during the academic year, a series of interviews and observations occurred in conjunction with the subjects' *Computer-Intensive Algebra* classes[1]. Each series consisted of one to three observations of the subject's *Computer-Intensive Algebra* class and both a pre-observation and post-observation interview. Two series were also conducted of one subject's non-*Computer-Intensive Algebra* classes. Each series provided information on the subject's instructional decisions, interpretation of interactions with students, and knowledge of the computing tools and mathematics. Observers and interviewers audiotaped the exchanges and took extensive field notes. Annotated transcripts of both the observations and the interviews were produced for analysis.

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[1] In addition to the authors, Sarah Eaton and Pete Johnson conducted observations and interviews which comprised the observation-cycle data for one of the subjects.

Analysis

The data are currently being analyzed for evidence of teacher empowerment, including evidence of teacher growth in abilities in two overlapping areas: understanding how to use technology to study mathematical concepts, and understanding student mathematical understanding, assessment and pedagogy in technology-intensive environments.

Three key categories arose with respect to teachers' own use of technology to learn mathematics and on their flexibility in their use of that technology. The first is their use of the tool for mathematical purposes. Each subject developed a singular approach to achieving each of their mathematical goals with the tool; novel approaches were rare. Their use of the tools for mathematical purposes related closely to their observed level of awareness of the architecture of the tool. The subjects did not express or demonstrate an awareness of superstructure. It seems that their lack of demonstrated awareness of the tool's architecture could explain why they did not capitalize on that structure to use the tool flexibly.

In addition, the subjects seemed have particular expectations related to the appearance of the computer display that minimally, if at all, related to mathematics relevant to their exploration of mathematical ideas and critical to their efficient yet flexible use of the tool in their explorations, demonstrations, and discussion of the technology. Their level of concern about format seemed to affect their choices of strategies to investigate ideas and solve problems. Tool-based interactions with students as well as with other teachers seemed to encourage growth with respect to mathematical use of tools, awareness of structure, and role of format.

Our analysis of teachers' understanding student mathematical understanding in technology-intensive environments involved teachers' perceptions of how students learn mathematics in these environments, their fluency in engaging students in doing and learning mathematics rather than only studying others' mathematical conclusions, their use of students' understanding in designing and implementing classroom instruction, and their choice and execution of various forms of assessment. The subjects' views of learning did not seem to be tailored to specifically to technology-intensive learning. In many ways they seemed to apply directly the perceptions of learning that had served them in classes which had not used technology.

One subject's structuring of activities that involved student exploration attempted to reflect a view of learning as discovery. However, that discovery was viewed more as a prescribed activity than as a guided activity. Even though the subject may have believed that students need to understand mathematics on their own, the subject's actions often suggested that it was the teacher who had the responsibility to help students acquire that understanding. Truly open-ended explorations, ones in which students determine to a great extent the direction of the exploration and develop understandings that may differ from those of their peers, conflict with a view of learning that puts responsibility for students' understandings with the teacher rather than with the student.

A predominant theme that emerged in our observations centered on what may have been conflicting views of teaching and learning held by each of the subjects. The subjects may have adopted views of teaching that included a commitment to construct classroom environments in which students worked in groups or pairs, to eliminate "teaching as telling" from their teaching repertoire, and, in one case, to focus on increasing the open-endedness of classes. This view of teaching may have been in direct conflict with the residue of a mastery view of learning which suggested that all students must master each section before the class could move on to the next section of the text. Students should explore mathematics instead of listening to teachers tell about it; but if the teacher were not going to 'tell' the answers, how were students to master the material? Even though students may work on exploring mathematical situations, in one case the subject seemed to view public statement and acknowledgment of conclusions as necessary and sufficient for students to accept and understand those conclusions. Our observations also suggested a belief by the subjects that student understanding could only be shown through the completion of mathematical tasks. The result was that, frequently, the subjects seemed to focus assessment on whether the students could complete certain tasks accurately rather than on the nature of the understanding they were developing. In this way, instruction as well as assessment seemed to be defined by the text in each case.

Other current working hypotheses center on: teachers' tendencies to focus on the technical aspects of the technology instead of on the mathematical ideas that can be explored through its use; the tension between traditional teaching techniques and the demands of a computer-intensive classroom (e.g., teachers' tendencies to apply a one-minute per student principle to their interactions with *Computer-Intensive Algebra* students even if that one minute does not give them a complete picture of the students' understanding or work). We are continuing our analysis as we play one data base off another.

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PERCEIVED DEFICITS IN MIDDLE GRADES MATHEMATICS TEACHING[1]

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This paper is a report of an extensive survey study of the practices, beliefs, and preparation of middle grades mathematics teachers. It addresses the relationships between what middle grades mathematics teachers say they do and what they believe they should do. In addition, it compares these responses with those of middle grades principals, mathematics curriculum coordinators, and experts in the area of middle grades mathematics. Results suggest that the discrepancy between middle grades mathematics teachers' practice and their beliefs is not entirely explainable by their lack of understanding of and preparation for what they should be doing.

The NCTM *Standards* (1989) and the MAA *Call for Change in the Mathematical Preparation of Teachers* (1991) propose a new vision of mathematics teaching at the middle grades level. The feasibility of implementation rests on the extent to which the personnel in schools are able and willing to act accordingly. There has been considerable research on the relationships between beliefs and practices of mathematics teachers. Often these studies have concentrated on comparing classroom observations of a particular teacher with that teacher's professed beliefs about the teaching of mathematics. These studies have yielded discrepant results about the relationship between what a teacher believes about the teaching of mathematics and his or her actual classroom practices. Some studies suggest a congruence between teachers' beliefs about mathematics teaching and their actual instructional practices (Grant 1984; Shirk 1973), while others demonstrate a considerable discrepancy between beliefs and practice (Brown 1985; Cooney 1985; Shaw 1989; Thompson 1984). What is not clear from these studies is the extent to which teachers (or principals or mathematics curriculum coordinators (MCCs)) think that mathematics teaching practices match beliefs about the teaching of mathematics.

This is a study of "perceived deficits" in the teaching of mathematics at the middle grades levels. "Deficit" refers to the mismatch between what teachers say they are doing and what they believe should be done as well as to the mismatch between the beliefs of experts on the NCTM *Standards* and the current beliefs of teachers and other school personnel. The study surveyed middle grades teachers, principals, and MCCs throughout a large Eastern state concerning their beliefs about what is and what should be regarding content, pedagogy, and structure of middle grades mathematics. Respondents were not individually identifiable but representative of a variety of school types (middle school, elementary school, and junior high school), demographies (town, rural, urban, and suburban), and (in the case of teachers) certifications (secondary mathematics and elementary). A consensus of expert responses was also determined by using a Delphi method with a panel of experts (the authors of the NCTM Middle Grades *Standards*).

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[1] This study used a series of survey instruments developed by Susan Jane Feeley and Lois Ostrowski under the direction of M. Kathleen Heid.

The survey

The survey had three versions (Teachers, Principals, and MCCs). The teacher survey was designed to compare what teachers said they were doing in their mathematics classrooms with what they said they thought they should be doing and with the additional preparation they said they needed in order to teach middle grades mathematics to their satisfaction. The principal and curriculum coordinator surveys polled these groups on what they thought their teachers were doing, on what they thought their teachers should be doing, and on what additional preparation they thought teachers should have.

Items for the survey were designed to reflect the suggestions offered in the NCTM *Standards*, in the MAA *Call for Change*, and in the expository literature on middle grades education. The set of survey items was critiqued for structure, ambiguity, vocabulary, and representativeness and reflection of the Standards documents both by a survey expert and by a panel of Mathematics Education faculty and graduate students. Through mail and in person, the survey was piloted to ascertain how well the survey items would be understood by potential respondents, and subsequently revised. Items used on the final survey polled middle grades mathematics teachers on the extent to which they included activities related to specific mathematics content areas (algebra, functions, and patterns; by-hand computation; connections and real world applications; geometry and measurement; number concept, estimation, mental arithmetic, and number sense; data analysis, probability and statistics; problem solving; conjecturing and reasoning) and to specific teaching strategies and structural arrangements (assessment and diagnosis; communication; manipulatives and instructional aids; individualization; and technology), and to current mathematics education issues (mathematics for all students and tracking). Principals and curriculum coordinators responded to similar surveys about the middle grades mathematics teachers with whom they worked.

The survey was mailed to 2197 middle grades teachers, of whom 288 responded. The response rate was not unexpected given the length of the survey (120 items) and the fact that the anonymity of the survey precluded follow-up reminders. Somewhat better response rates were obtained from the principal and mathematics curriculum coordinator surveys.

Results

Each of the 120 items on the survey yielded specific information about the relationships between reported practices and reported beliefs for teachers, principals, and mathematics curriculum coordinators, and among all three groups. A complete item analysis of the survey would be impossible to convey in a short amount of space. Instead, we will give one example of a type of item analysis that was possible.

Calculators in middle grades mathematics classrooms

This example focuses on use of calculators in middle grades mathematics classrooms. Summaries of responses from all three groups to a sample item on the use of calculators in middle grades classrooms from the "practice - belief" section of the survey are shown in the following charts. The asterisk indicates the answer or answers agreed upon as appropriate by the panel of experts. In this case the experts indicated that middle grades mathematics students should use calculators in class almost daily.

The response summaries to these items show that small percentages of the respondents (19% of the teachers, 20% of the principals (with another 15% declining to comment), and 36% of the mathematics curriculum coordinators) agreed with the experts that students should use calculators in almost every mathematics class. In fact, a large portion of each group (57% of the teachers, 38% of the principals, and 29% of the mathematics curriculum coordinators) felt that calculators should be used in at most a moderate number of classes. As shown in the following set of charts, the discrepancy between the experts and the respondents was even greater when it came to use of calculators during testing. The response summaries to these items show that even smaller percentages of the respondents (14% of the teachers, 9% of the principals, and

Teacher item 17. In my mathematics classes, students use calculators.

Teacher Item 17	1	2	3	4	* 5
Practice	25.69	19.79	30.56	13.54	10.42
Belief	7.17	16.49	33.69	24.01	18.64

Mathematics curriculum coordinator item 17. In my school district, middle grades mathematics students use calculators.

Mathematics Curriculum Coordinator Item 17	1	2	3	4	* 5
Practice	15.52	29.31	29.31	17.24	8.62
Belief	0.00	1.79	26.79	35.71	35.71

Principal item 17. In my school, middle grades mathematics students use calculators.

Principal Item 17	1	2	3	4	* 5	NC
Practice	12.50	27.50	22.50	22.50	15.00	0.00
Belief	2.50	15.00	20.00	27.50	20.00	15.00

Percent of respondents choosing each response option for selected calculator item on the Middle Grade Mathematics Project surveys.

Key
1 Almost never
2 Occasionally
3. In a moderate number of classes
4. In a majority of classes
5. In almost every class

25% of the mathematics curriculum coordinators) agreed with the experts that students should always have access to calculators on mathematics tests. Moreover, a large portion of each group felt that calculator use should be moderate at most on tests (67% of the teachers, 56% of the principals, and 39% of the mathematics curriculum coordinators). If one were to take the recommendations of the panel of experts as goals for the teaching of middle grades mathematics, one can see several indications of difficulties in implementing those recommendations in the case of classroom calculator use. First, middle grades teachers are far from meeting the goal of providing students with universal classroom access to calculators (only 10% now provide such access in class and 7% provide such access on examinations). Second, most middle grades mathematics teachers do not now believe that they should provide students with universal classroom or testing access to calculators (81% do not believe in universal classroom access to calculators and 86% do not believe in universal examination access to calculators). Third, it is not probable that teachers will get strong support from administrators to implement universal

Teacher item 43. In my mathematics classes, I allow the use of calculators during evaluation.

Teacher Item 43	1	2	3	4	*5
Practice	36.46	24.65	20.14	12.15	6.60
Belief	17.14	25.71	24.64	18.93	13.57

Mathematics curriculum coordinator item 43. In my school district, the teachers of middle grades mathematics students allow the use of calculators during evaluation.

Mathematics Curriculum Coordinator Item 43	1	2	3	4	*5
Practice	34.48	31.03	20.69	10.34	3.45
Belief	5.36	5.36	28.57	35.71	25.00

Principal item 43. In my school, the teachers of middle grades mathematics students allow the use of calculators during evaluation.

Principal Item 43	1	2	3	4	*5
Practice	17.50	37.50	22.50	12.50	10.00
Belief	0.00	28.13	28.13	34.38	9.38

Percent of respondents choosing each response option for selected calculator item on the Middle Grade Mathematics Project surveys.

Key
1 Almost never
2 Occasionally
3. In a moderate number of classes
4. In a majority of classes
5. In almost every class

classroom and testing access to calculators. Although principals and mathematics curriculum coordinators are closer to agreeing with the experts on this issue, there is still a vary large percentage who do not support the experts' recommendation.

When, on a different section of the same survey, teachers were asked whether they would like more preparation in the use of calculators in teaching mathematics (and when principals and mathematics curriculum coordinators were asked whether their teachers needed such preparation), there was a large amount of support for additional preparation. Sixty-seven per cent of the teachers indicated that they would like more preparation in this area. Principals (66%) and mathematics curriculum coordinators (85%) agreed. Of the 46 items on the "preparation" section of the survey, only five had greater support among teachers. One might surmise that, with additional preparation, beliefs and practice might change in the direction recommended by the experts.

General trends in beliefs and practices about middle grades mathematics

Numerous statistical tests were performed comparing the responses of elementary and secondary certified teachers, principals, and mathematics curriculum coordinators (Feeley & Heid 1993). Among the general conclusions resulting from these analyses of the survey results are the following:

1. What middle grades mathematics teachers report doing in teaching mathematics differed significantly from what they say they think they should do on 88% of the items. This was true no matter whether the teachers were certified as secondary mathematics teachers or as elementary teachers. Similar results occur for the principals and MCCs.

2. The beliefs of teachers, principals, and MCCs regarding what is and should be happening in middle grades mathematics classrooms do not match those of the panel of experts for the vast majority of items. Although all three groups have beliefs that are far from what was agreed upon as acceptable by the panel of experts, the beliefs of MCCs are closest to those of the experts and the beliefs of the principals are furthest from those of the experts. Principals' responses also suggest that they are more satisfied with the classroom practices of their teachers than are the teachers as a group or the MCCs.

3. Teachers are reporting classroom practices that reflect, when compared to the experts' recommendations, too little attention given to: nonroutine problems, alternative solution strategies, and original solutions; statistical concepts, and work with collecting , analyzing, and reasoning about data; geometric and algebraic patterns and relationships; real world applications; number sense, mental arithmetic, and applications; use of technology; and connections between algebraic and geometric ideas. Teaching strategies that receive too little attention include: small group work, having students write about mathematics, and using manipulatives to enhance conceptual understanding.

This study suggests that middle grades teachers (both elementary and secondary mathematics certified) recognize a discrepancy between what they think they ought to be doing and what they report that they are doing in the teaching of mathematics. In addition, although their beliefs are closer to those of the experts than what they report doing, in a vast majority of cases there are significant differences between what they report believing they should do and what the experts suggest should be done. Further work should examine the reasons for these differences. Potential sources for these discrepancies suggested in the survey are lack of preparation, lack of support, and lack of appropriate institutional policies and structures.

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HOW MIDDLE SCHOOL TEACHERS ADJUST TO CHANGE: CLASSROOM TESTING OF MATERIALS FROM AN INNOVATIVE CURRICULUM PROJECT

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Twenty-six sixth and seventh grade teachers who were involved in field testing an innovative middle school curriculum during 1993-94 were studied to identify and analyze their concerns as they adapted to using the new curriculum. Data were collected at the beginning, middle, and end of the school year using the Stages of Concern questionnaire (SoCQ) (Hord, Rutherford, Huling-Austin, and Hall, 1987), Additional data about the teachers were available from classroom observations, personal interviews, and other questionnaires completed by the teachers and their students. Data show that as a whole teachers progressed through the stages of concern much as would be expected, but there was a subset of teachers for whom scores were initially surprising.

Being concerned about change is universal, whether the change is associated with our personal lives or with our work. In particular, teachers experience both individual and common concerns when they attempt to make changes in their approaches to teaching or in the curricula they use. Concerns exert a powerful influence on the implementation of a change, and they also determine the types of support and assistance that teachers find useful. This study was designed to identify and analyze the concerns experienced by a group of teachers from around the country as they field tested an innovative middle school curriculum.

The Connected Mathematics Project (CMP) of Michigan State University, with funding from the National Science Foundation, is developing a complete mathematics curriculum for grades 6, 7 and 8. Writing and pilot-testing of the CMP materials began in 1991-92, and will continue for a total of five years. Teachers and students using the CMP curriculum are engaged in teaching and learning quite different from the traditional transmission/reception mode since the project is committed to developing materials that engage students in learning mathematics through contextualized "investigations"— activities that involve groups of students in active involvement with mathematical concepts and applications and in reflective writing and talking about these ideas.

Evaluation data for CMP are being collected from classrooms in California, Michigan, New York, North Carolina, Ohio, Oregon, Pennsylvania, and Washington. The 12 CMP field testing sites were chosen to represent a diversity of academic, socioeconomic, and ethnic backgrounds. During the 1993-94 school year, a variety of data were collected from some 30+ sixth and 20+ seventh grade teachers who had agreed to field test the entire year's curriculum. An individual was hired at nine of the field testing sites to serve as recorder — that is, to observe weekly in some of the teachers' classrooms; to administer project-developed questionnaires to teachers and students three times during the school year to collect information about their attitudes toward change, their beliefs and attitudes about mathematics and mathematics teaching, and their reactions to the CMP materials; and to conduct periodic interviews with select teachers and with a subset of their students so that profiles of these individuals could be developed. The recorders also collected examples of teacher plans and student work from some classrooms as well as vignettes of classroom learning events and classroom conversations.

Although a number of broad purposes have been identified to frame the CMP evaluation effort, the purpose of this paper is to focus attention on the 1993-94 teachers' movement through the process of adaptation to change. Research has identified seven stages of concerns that users of an innovation may have (Hord, Rutherford, Huling-Austin, and Hall, 1987). These stages are identified in Table 1, which also gives an example of each type of concern.

Table 1. Stages of Concern and Sample Expressions of Concern

Stages of Concern	Sample Expressions of Concern
6 Refocusing	I have ideas that might work even better than this new curriculum.
5 Collaboration	I'm concerned about how I can relate what I"m doing with what other teachers are doing.
4 Consequence	How is the new curriculum affecting the students?
3 Management	I seem to be spending a lot of time planning and organizing.
2 Personal	How will using the new curriculum affect me?
1 Informational	I'd like to know more about the new curriculum.
0 Awareness	I don't even know what the new curriculum is all about.

The Concerns-Based Adoption Model (CBAM) groups the seven stages into three dimensions—*Self* (Stages 0-2), *Task* (Stage 3), and *Impact* (Stages 4-6)—that are considered to be developmental in nature. That is, when efforts to change are in their early stages, teachers are likely to experience *self concerns* (wanting to know more about the innovation and how it will affect them personally). *Task concerns* are more common as final preparations are made for starting an innovation and in the early phases of its implementation. When teachers have begun to worry about the effects of the innovation on their students and about what can be done to improve its effectiveness, then they are experiencing *impact concerns.*

Data concerning teachers' concerns about using the CMP curriculum in their classrooms were collected by administering the Stages of Concern Questionnaire (SoCQ) (Hord, et al., 1987) to CMP teachers at the beginning, middle, and end of the 1993-94 school year. The SoCQ, a 35-item paper-and-pencil measure developed through extensive research that has assured its validity and reliability, can be used to produce profiles showing the intensity level of an individual's concern for each of the seven stages of the CBAM model. This paper focuses on the responses of the 26 CMP teachers (of about 50) who had returned all three SoCQ questionnaires at the writing of this paper.

Since the CBAM model sees change as a process, rather than an event, individuals are almost certain to have some degree of concern from all stages at any given time, although the stage or stages where concerns are focused generally tends to vary as the change progresses. Other implicit assumptions in the model are that the individual must be the primary target for change, that change is highly personal, and that stages of change involve individuals' feelings about the innovation as well as their skill in its use (Hord, et al., 1987).

These assumptions led us to ask a number of questions about the 26 teachers in this study (all of whom were field testing CMP materials in 1993-94). These teachers included 10 who had previously pilot tested draft versions of CMP materials for one or two years, and 16 teachers who were new to CMP in 1993-94. We were interested in determining which concerns were highest for CMP teachers, and in describing how concerns of the group changed over the course of the year, as well as in trying to profile individuals whose concerns either seemed extreme or changed radically during the year. According to the CBAM model, we expected that the new teachers' concerns would more likely be with self and task than with impact, although we were uncertain what types of concerns to predict for the returning teachers since they already had some familiarity with the CMP philosophy and materials, but were nonetheless trialling the 1993-94 versions of the materials sight-unseen. In addition, we wondered if experienced teachers would be likely to move more quickly than novice teachers from self to task concerns, and then to impact concerns.

By combining information from the SoCQ with data collected by the recorders, we hoped to be able to describe how and to what extent stages of concern about the use of CMP materials might be related to changes in the culture of the mathematics classroom such as the nature of learning tasks, the organization of students for learning tasks, and the nature of classroom assessment. We were also interested in studying factors that seem to shape the ways the CMP materials are used. Factors that seem to be emerging include teacher subject matter knowledge (Hashweh, 1986; Kuhs & Ball, 1986); teacher beliefs about mathematics and what it means to know mathematics; teacher familiarity with the pedagogical style promoted by the CMP materials; teacher and school expectations for students; and contextual elements such as district objectives and testing programs.

The data from the 26 sets of questionnaires were analyzed in several different ways, including an examination of changes in the entire set over the year, comparisons between new and returning teachers, study of subgroups of individuals with mostly high levels of concern or mostly low levels of concerns, and an investigation of individual cases or small groups of cases that seemed noteworthy. Space constraints allow only a brief overview of the findings from these analyses here, but more details will be presented at PME.

As expected, teachers showed movement through the stages of concern throughout the year. The following table shows how many of the 26 teachers marked various stages as their top concern at each of the three administrations.

Table 2. Top Teacher Concerns by Administration (n = 26)

Top Stage of Concern	Pre-Year	Mid-Year	End-Year
6 Refocusing	0	2	5
5 Collaboration	6	7	8
4 Consequence	8	3	4
3 Management	1	6	4
2 Personal	2	0	1
1 Informational	2	3	1
0 Awareness	7	5	3

As can be seen, the teachers, as a whole, moved from lower to higher stages of concern from pre- to mid- to end-of-year. Before the start of the year, 11 teachers had top concerns in the self dimension (Awareness, n=7; Informational, n=2; and Personal, n=2), one teacher's leading concern was in the task dimension (Management), and 14 were most concerned with stages in the impact dimension (Consequence, n=8; Collaboration, n=6; Refocusing, n=0). By year's end, only five had top concerns in the self dimension, four were most concerned with stages in the task dimension, and 17 manifested top concerns in the impact dimension.

Table 3 shows a comparison of new and returning teachers. For the sake of brevity, only beginning and end-of-year data are used. Numbers shown are percentiles, based on extensive research with the SoCQ (Hord, et al., 1987).

Table 3. Mean Concerns Percentiles of New and Returning Teachers (Pre-Year and Post-Year, n=26)

Stages of Concern	New (Pre)	Returning (Pre)	New (Post)	Returning (Post)
6 Refocusing	29	35	50	45
5 Collaboration	60	71	70	65
4 Consequence	59	57	51	64
3 Management	48	35	53	46
2 Personal	54	38	39	38
1 Informational	58	33	44	25
0 Awareness	60	52	51	32

As can be seen from Table 3, at the beginning of the year, new teachers had, on average, higher scores than returning teachers in the self dimension, (as would be expected). Also, new teachers were somewhat lower on the impact dimension than returning teachers. By the end of the year, concerns among the new teachers had shifted quite a bit; concerns at the self level had dropped and most of those in the task and impact dimensions had increased. In fact, progress through the stages of concern for the new teachers was greater than that of the returning teachers, although one might argue that at the end of the year the overall concerns profile for the returning teachers was "more advanced" than that of the new teachers (based on lower levels of concern at the awareness and informational levels). It is interesting to note that at the end of the year mean concerns for the new teachers were higher than mean concerns for returning teachers on six of the seven stages.

Looking at individual profiles in more detail, we were intrigued to note three teachers whose mid-year profiles were somewhat similar, although rather different from most of the others. Each was quite experienced with mathematics education innovations: one as a CMP teacher the previous year, and as a writer of CMP assessment materials; the second as a CMP recorder the year before, and as a teacher and inservice trainer with several other innovative curricula; and the third as a CMP pilot teacher from the year before who had also assisted in writing portions of the curriculum. The interesting development was that each had a rather high score on the management stage of concern on the mid-year questionnaire. Their percentile scores were 80, 95, and 83 respectively. While these scores would not be overly high for some participants (since the curriculum does place considerable management demands on teachers), these three teachers were already quite familiar with the curriculum before the year started. At that beginning point, the first and second teachers both scored only in the 23rd percentile on management; the third teacher completed neither an initial nor a final survey and therefore cannot be included in the full discussion, regrettably since her mid-year profile was so similar.

What made these experienced teachers rate management concerns so high at mid-year and then still rate them among their highest concerns at the end of the year (56th and 85th percentile, respectively)? Our best hypothesis, confirmed in part by followup conversations with them, is that their management concerns were high because they were sincerely attempting to implement the curriculum fully, including detailed attention to the content of the materials, the methodology suggested, and the assessment tools provided. This is not to say that other teachers were not attempting to teach the curriculum as intended. However, the recorders' reports indicate that very few of the field testing teachers were able to manage full implementation (see Doyle & Ponder, 1977 for a discussion of how teachers tend to adapt, rather than adopt, innovations). In particular, assessment practices recommended by the curriculum (e.g., partner quizzes, journal writing, and portfolios) were not always used by teachers who otherwise tried to follow content and pedagogical suggestions.

A closer look at the individual items that contribute to the management score may be enlightening. The teacher with the management score in the 56th percentile marked three of the items with a five (on a scale from zero=irrelevant to seven=very true) and the other two zero. The items marked five were

4. I am concerned about not having enough time to organize myself each day.
16. I am concerned about my ability to manage all the innovation requires.
34. Coordination of tasks and people is taking too much of my time.

The two items which were marked zero were

8. I am concerned about the conflict between my interests and my responsibilities.
25. I am concerned about time spent working with nonacademic problems related to this innovation.

Thus, this teacher's high management score was very much a result of concerns with classroom time demands in using the innovation, not more peripheral concerns. This breakdown of the management category reinforces the idea that this teacher is wrestling with the demands of teaching an innovative curriculum in an investigative manner and then assessing student work in a manner consistent with the instruction.

Findings from studies like this can help in the design of inservice workshops to familiarize teachers with new curricula. Hord, et al. (1987) provide some suggestions for interventions that might be useful in responding to the various levels of concern that teachers may have (e.g., provide answers that address specific "how-to" issues in management concerns). An introduction to the curriculum, if well done, should move teachers rather quickly through all three beginning stages—Awareness, Informational, and Personal—so they can focus their attention on implementation issues as they begin using the materials with their students. Although it is desirable for teachers to be concerned with Consequences (that is, with how a new curriculum will affect their students) and with Collaboration (how they can work in concert with other professionals), teachers generally must bring their levels of concern about self and task (i.e., Management issues) under control before they can focus their attention on issues of impact (which includes Consequences and Collaboration). As one might expect, teachers are generally not concerned about Refocusing (modifying, revising, supplementing, or enhancing the curriculum) until they have the new curriculum itself under control. This helps explain why collecting feedback from trials of a new curriculum is so problematic. As long as teachers are struggling with self and task issues they are unlikely to be able to provide much useful feedback about potential revisions to the curriculum.

A considerable amount of work remains to be done with the SoCQ data. In the future we intend to develop profiles of teachers who have consistently high scores on five or more stages of concern and teachers who have consistently low scores on five or more stages, to investigate teachers whose profiles change drastically over the course of a year, to pursue concerns profiles of caricatures developed previously (Lambdin & Preston, 1993), and to follow the group of teachers discussed in this paper through an additional year of use of CMP materials.

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THE NEEDS OF SECOND CAREER, SECONDARY MATHEMATICS TEACHERS: HOW WELL ARE THEY MET BY ACADEMIC PROGRAMS AND INDUCTIVE PROCESSES?

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SUMMARY

This naturalistic study investigates the how well the needs of second career mathematics teachers are met by obtaining data from a series of three in-depth interviews and two written response documents, exploring the current traditional academic certification programs at three universities, and investigating two specialized programs: a military career transition program and a state's alternative route to certification. The five case studies and cross case analysis examine the respondents' motivational, learning and academic needs. Implications discovered indicate that programmatic adjustments and enriching curriculum will help strengthen existing programs and better satisfy the needs of these neglected pre-service second career mathematics teachers.

THEORETICAL FRAMEWORK

During the past ten years, the number of individuals seeking career changes has increased dramatically (Lacey, 1988; Herr & Cramer, 1988). With reasons rooted in the changes in society, personal circumstances, and their need for increased perception of self worth, many of these adults, who have already obtained a baccalaureate degree, desire to become educators (Bliss, 1990). By exploring and addressing the needs of these adults, insights into improving the quality of traditional, special, and alternative routes of certification, and institutional and inductive support models are discovered.

The several different sets of needs examined are categorized into three major areas: motivational, learning, and academic. Motivational needs, such as security, affiliation, self-esteem, autonomy and self-actualization are linked to life's stages. The accepted motivational theories: Maslow/Porter's hierarchy-of-needs theory (1961) and Herzberg (1966) two-factor theory of motivation provide a theoretical framework. The learning needs, which differ from those of the traditional student, are examined with emphasis on the Andragogical model (Knowles, 1984). The academic needs, such as introducing the person to recent technological advancements and updating his or her mathematics skills, are addressed.

The emergence of post-degree students in pre-service teacher education programs occurred when widely publicized national reports predicted that America was soon to face a severe shortage of teachers (Madison & Hart, 1990). The forecast for qualified mathematics teachers was even more direful, indicating that definite factors exacerbated the shortage of qualified teachers (NRC, 1986). At the same time, state departments of education and institutions of higher learning initiated a focused interest in teacher education (Holmes Group, 1986; Carnegie Task Force, 1986).

Additionally, the mathematics education community prepared documents which echoed the challenge that mathematics teachers be better prepared. Focusing on what the teacher needs to know about mathematics, mathematics education and pedagogy to implement the teaching vision, these recommendations provide guidance to both colleges and universities as they develop their mathematics educational programs and to state departments as specialists establish certification requirements for mathematics teachers (NCTM, 1989, 1991).

Alternative certification represents a rapidly evolving new route for post-baccalaureate individuals to enter the teaching profession. This preparatory certification route allows candidates who are subject matter specialists to circumvent the traditional pedagogical learning which generally lasts two years. Although each program differs slightly, all carry an intensified period, generally summer when the candidates learn teaching skills. This period is followed by an internship program within the classroom where the candidate performs classroom duties under the guidance of certified teachers (Feistritzer & Chester, 1991; Zumwalt, 1991).

With an increasing number of second career individuals becoming mathematics teachers who possess different educational needs, views, and qualities than their 22 year old counterparts, teacher preparation and induction should be addressed. The efforts to develop more effective teachers can enable students to obtain mathematical power envisioned as desired for the 21st Century living.

METHODOLOGY

Through this naturalistic inquiry the researcher seeks to illuminate the perceptions and social realities of the five individuals, purposefully selected, by gathering a thick description. Since qualitative inquiry is not designed to be generalizable, the inquiry establishes existence of issues not previously seen. As a highly interactive method of gathering data with the simultaneous analysis, this naturalistic study began with the research question being formulated. Contextual answers were sought by conducting a series of three guided interviews with each of the five second career, secondary mathematics teachers. Each respondent:

1) Is a second career secondary mathematics teacher who has been teaching no longer than 5 years.
2) Has been in non-teaching career path for at least five years.
3) Has previously earned a bachelor's degree in mathematics, a related field of study, or has taken additional mathematics courses for secondary mathematics certification.

The first interview started with one prepared question: "Can you tell me something about your background which helped you to make this career choice?" Exploration questions were asked during the first phase of the study, which consisted of two interviews for each respondent. This research approach was simultaneously accompanied by checking accuracy, probing, and confirming data through additional questions and other triangulation methods. The second phase involved analyzing categories of topics explored during each respondent's first two interviews. During the third phase the final interviews were conducted to make certain that the same topics were queried. During this phase the analysis time increased with the decrease of data collection time.

Planning for trustworthiness remains a priority of qualitative research, with time and effort representing key ingredients. *Credibility*, which is compared to *Internal Validity*, was accomplished through peer debriefing, a three month engagement with the respondents, triangulation through requested written documents, and member checks and negotiations. *Dependability*, as compared to *Reliability*, was established through a methodological log, and the researcher's personal reflective journal. *Confirmability*, as compared to *Objectivity*, uses raw notes, records of meetings, the needs outline used in the analysis, and decision rules used in categories established in the analysis (Lincoln & Guba, 1985).

CASE STUDIES AND CROSS CASE ANALYSIS

The five second career secondary mathematics teachers in this study represent novice teachers who have taught from two to five years and instruct students in a wide variety of mathematics: seventh grade topics through calculus. Four of the five teachers attended institutions of higher learning where certification programs enabled them to gain pedagogical knowledge; the other teacher attended a summer alternative certification program. Three of the teachers, with pseudo names Bill Thompson, Henry [Hank] Flanders and Linda Barrington, are teaching in senior high schools. The two others, Paula Andrews and Ruth Harrison, teach in middle schools. The case studies provide a portrait of each person as a unique, holistic entity, by describing what it was like to be in each of the various programs and to teach secondary mathematics. Their motivational, learning and academic needs were met in varying degrees

Motivational Needs

This study clearly supports the hierarchy-of needs hypothesis presented by Maslow/Porter (1961). Bill, Linda, and Paula seem to be in higher levels, perhaps actualizing there abilities sooner than most first career teachers with an equal number of teaching years. Likewise, Ruth and Hank, who are experiencing teaching difficulties, are in the lower two levels. This information supports the Herzberg two-factor theory of motivation (1966), which applies to maintenance and motivation factors.

Learning Needs

The interviews appear to confirm the findings of those researchers who regard adult learners in higher education as highly motivated, time-oriented, self-critical, and aware of themselves, their careers, and what they want to accomplish. The most striking elements are their positive, goal-centered need to learn and their outspoken personality characteristic. All desire very practical learning experiences, rather than theoretical concepts and generalizations. They possess little toleration toward bureaucratic situations and desire to achieve success both academically and personally, through short-term goals.

Academic Needs

These individuals generally deal positively with the personal concerns by establishing their support network, and dealing creatively with financial and time constraints. The teachers who became certified by the traditional programs were introduced to much theoretical pedagogical knowledge, but all desired more concrete, practical examples of how to put this theory into practice in the classroom. In contrast, the alternative certification individual was introduce to a broad base of educational techniques from practitioners, but disappointingly she was not introduced to the theoretical underpinnings through research from mathematics education. All novice teachers interviewed lacked the practical methods of dealing with classroom discipline in a positive manner. The learning of timely interpersonal skills was conspicuously absent in their certification programs. In general, these respondents are not adequately prepared in the areas of mathematics teaching philosophy, technology, working with exceptional students, and multiculturalism.

CONCLUSIONS

These second career individuals not only seek to work with young people to promote positive learning experiences, but also recognize that teaching provides opportunities for their own personal growth. Suggested implications include: (1) Specially designed programs satisfied the recipients' needs better than traditional programs, (2) Formal mentoring programs help second career individuals adapt to teaching, (3) Learning and implementing classroom management techniques play a key role in self-esteem, (4) Positive attitudes combined with personality traits are major factors in the successful transition to teaching, (5) Confidence in mathematical content materials allows the teacher to concentrate on pedagogical techniques, (6) Familiarity with the current mathematics education philosophy enables the second career teacher to grow in teaching expertise, and (7) The respondents need better preparation in the areas of technology, exceptional students and multicultural education. Recommendations which acknowledge the current mathematics educational reform efforts and recognize novice teacher problems can be implemented through programmatic adjustments and enriched curriculum which will strengthened teacher preparation programs.

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DEVELOPMENT OF CLASSROOM SOCIAL NORMS AND MATHEMATICAL PRACTICES WITH PRESERVICE TEACHERS

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As part of a three-year study of the mathematical and pedagogical development of prospective elementary teachers, author Simon taught a mathematics course in which he engaged traditionally-educated undergraduates in renegotiation of the meaning of mathematical activity. In this report, we describe the non-traditional mathematical practices that emerged through the social interactions of this classroom community. We analyze excerpts from a discussion that occurred in this course, and attempt to portray some of the ways in which teacher and students interpreted each other's activity. This line by line analysis illustrates some routine teacher behaviors that encouraged the prospective teachers to view themselves as initiators of mathematical exploration and validators of mathematical ideas.

This research report presents one piece of a much longer paper. The longer paper examines the process by which a reform-oriented teacher educator and traditionally-educated prospective elementary teachers[1] negotiate norms for the practice of mathematics in their classroom. The work is part of a three-year study[2] of the mathematical and pedagogical development of prospective elementary teachers. As part of this study, the second author, Simon, taught two reform-oriented mathematics education courses conducted as constructivist teaching experiments (Cobb & Steffe, 1983; Cobb, Yackel, & Wood, 1993), one in mathematics (focusing primarily on multiplicative structures) and the other in mathematics learning and teaching. The vision of learning and teaching which guided both the content and process of these courses is fundamentally compatible with current reform efforts (NCTM, 1989; 1991).

This report focuses on the mathematics course and endeavors to portray and analyze the nature of the classroom mathematical practice that emerged. To do so, we examine excerpts from the fifteenth class session. Class 15 was chosen because it demonstrates many of the aspects of the evolving classroom mathematics tradition which differ greatly from traditional school mathematics traditions. The course included 25 1.5-hour sessions, in which the teacher created problem-solving situations to foster the construction of understandings of multiplicative structures. He asked probing questions, requested that ideas be paraphrased, managed and focused the discussion as needed, but avoided using his authority to indicate the correctness or value of particular ideas. The students worked on problems in small groups before discussing them with the whole class. (The teaching of this course is analyzed in depth in Simon (in press).)

We videotaped and transcribed each class meeting and collected copies of the students' weekly journal reflections and written work. We view the 26 students and their mathematics teacher as a classroom community with its own evolving mathematics tradition in the sense of

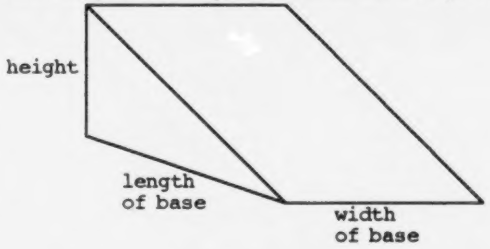
Kuhn (1970) and Cobb, Wood, & Yackel (1991). We interpret classroom interactions from the perspective of symbolic interactionism, in a manner similar to Bauersfeld, Krummheuer, & Voigt (1985) and Voigt (1985).

Class 15

By this time, the group had established certain norms of appropriate mathematical behavior. After briefly explaining the context of this discussion, we illustrate the community's ways of functioning together through analysis of excerpts from the class transcript.

Simon's goal for this section of the course was to foster an understanding of ratio as the measure of an attribute of a physical situation (see Simon & Blume, 1994) along with greater understanding of when multiplicative comparisons were appropriate as opposed to additive ones. The following problem was the second problem on a worksheet given to students.

> In Kansas, there are no mountains for skiing. An enterprising group built a series of ski ramps and covered them with a plastic fiber which permitted downhill skiing. It is your job to rate them in terms of most steep to least steep. You have available to you the following measurements for each hill: the length and width of the base (measured along the ground) and the height. See the diagram. How would you determine the relative steepness using the information that you have?



During several class sessions, the students worked on this problem and related problems in their small groups and alternately discussed their thinking in whole class discussions. The students perceived early on that steepness was increased by greater height and/or decreased length of base. The focus of the inquiry became whether the ratio of these quantities or their difference (subtracted) was an effective indicator of steepness. The students struggled with what it would mean for a method to "work." They had difficulty structuring their investigation so as to be able to draw conclusions (e.g., they chose seemingly random lengths for the height and base of their triangles, rather than comparing two triangles of same ratio or same difference). They also had difficulty determining what could constitute evidence for or against one of the proposed methods. An analysis of the understandings involved is included in Simon & Blume (1994).

In Class 15, based on the previous discussion, Simon posed the following problem for the students to work on in their small groups:

> Order the following slopes in terms of their steepness. Use any means about which you are sure it works. Then go back and explore our ratio and difference methods using these data: 3, 1 4, 6 11, 9 2, 6 9, 3 7, 11 12, 4

After students had worked on this problem in small groups for about 40 minutes, they had the following discussion.

T: All right. (pause) Would you raise your hand if you . . . persuaded yourself conclusively whether the ratio method works and you persuaded yourself conclusively whether the difference method works. . . . So who would like to present us with some persuasive arguments? (pause) Georgia.

Geo: We all drew these slopes, OK, and then we used the ratio method and what we found was three of the slopes were equivalent, OK, meaning that three of the slopes using the ratio method came out to be 3. So what that told me was that . . . using the ratio they could all be reduced down to 3, and they all had the same angle of steepness. But in the difference method if we would have subtracted 3 from 1 we would, we would have got 2, if we subtracted 9 from 3 we would have got 6, and they would have all been different, OK? So, and with the rest of the slopes we would have found negative numbers if we would have used the difference method. (short pause) We just can't have negative slopes. I don't know, maybe in Kansas they have negative slopes . . . It just doesn't make sense to me to have a negative slope.

Georgia points out that three of the pairs of heights and bases provided are equivalent by the ratio method and unequal by the difference method. In responding to Georgia's contribution, Simon directly addresses neither the group's conclusions about the ratio method, nor Georgia's comment about negative slopes. Instead, he turns the argument to the class for their consideration, indicating his expectation that the students are trying to understand and evaluate each other's arguments.

T: Judy, do you have a question about what Georgia said?

Jud: I agree with what she said, but um, I drew them all out, and you can see in those she used, three where you get the same slope for them, that they look the same, like the shape of them, they look the same. . . . But when you use the subtraction method, you get all different numbers, that's why I don't think that's accurate.

T: So when you looked at them . . . ?

Jud: I did the ratio method but then whenever I drew them all out, and I looked and you could see that those three all look [inaudible].

Judy seems to think that Georgia did not connect her information about slope using the ratio method with the pictures her group had drawn and this is the piece that Judy adds to the discussion. Judy shows that she believes her role is to make sense of Georgia's remarks and continue looking for clarification of the mathematics involved by adding to Georgia's ideas. Simon asks Judy several clarifying questions that simultaneously help the students understand her comments, and focus them on a contribution that he considers to be productive for the group's further understanding. ("Productive" does not imply "correctness.") He then turns her contribution back to the class.

Sara, Molly and Bobbie then offer a series of contributions that is no longer focused on which slopes are equally steep, but rather on the development of more persuasive and elegant (our language) methods of demonstrating equal steepness. Sara shows that embedding the triangles allows one to see equal steepness by observing the parallel hypotenuses (Figure 2A). Molly claims that she and Lois are "taking [Sherry's] idea and then doing it better." They demonstrate that embedding the triangles with the top vertices superimposed allows the observer to see equal steepness when the hypotenuses fall one on top of the other (Figure 2B). Bobbie sees the opportunity to generalize and adds that Sara and Molly have offered two of three possibilities; the triangles can also be embedded at the lower right vertex, providing a demonstration similar to Molly's (Figure 2C).

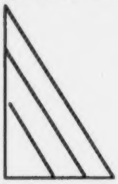


Figure 2A

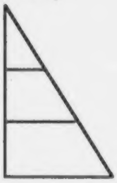


Figure 2B

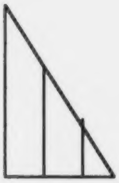


Figure 2C

This series of contributions indicates that, despite the group's understanding of Georgia's idea and agreement with Judy's point, the students do not think that discussion of the problem is over. Rather they see their roles as involving deriving the fullest understanding possible from the mathematical context.

Upon completion of this phase of the discussion, Simon brings the discussion back to what has been learned about the "ratio method" and the "difference method." As consensus as to the validity of each builds, Eve, Lilly, and Toni raise questions that delve further into the mathematics.

Eve: I have a question. . . . if we're doing the ratio method and you divide 9 by 3, OK, and you get your 3, what is that 3? What is it, is it the angle of degree, or what is it?

Eve wants a relation between the physical model and the calculations. The teacher praises her question, although he has not praised anything else in this discussion. Simon uses praise to support what he considers to be appropriate social norms, not to indicate correctness. He highlights the kinds of questions he wants the students to ask themselves and each other, thus teaching them how to think about mathematics. Although he helps to clarify her question, he makes no attempt to answer it himself and Georgia clearly does not expect him to, offering an answer immediately. Lilly, looking for greater understanding, suggests a possible connection between their mathematics problem and her own experience outside of class.

Lil: I was just wondering if that answer tells you, you know, when you go over the mountain and there's a truck stop and a sign that tells the trucks to gear down because you know, it's 7% slope or whatever, or I mean 7% grade. Do you all, have you seen those signs?

Lilly's comment indicates her expectation that classroom knowledge may connect with, even explain, some of her real world experiences and that such ideas are appropriate contributions to discussion in this class.

The discussion continues and Lois' response, referring to the result of taking a ratio, encompasses both Eve's original question and Lilly's speculation.

Loi: It's not telling you, like giving you a value for that side or telling you that this angle is this much, or this side is this long. What it's saying, it's sort of like a rate, OK, it's going, um, it's like a rate of steepness, it's sort of almost like a comparison, sort of.

Geo: If we had to look at it on a triangle on the board, where would you say that we're finding, where would that 3 go, in the triangle?

Loi: Think of like when...

Geo: No, you tell me where you think it would go.

Loi: It's not like that though. (short pause) It's not measuring any part of the triangle. It's measuring how much that triangle is tilting this way or that way sort of.

Geo: What determines that triangle to tilt this way or that way?

Loi: The relationship of the sides, how you draw them. Like if one side is 3 and, if it goes up 3 and over 1, OK, then that is going to determine those sides.

Toni brings up another question that delves into the deeper meaning of the ratio.

Ton: I was just wondering, is the number we come up with when we use the ratio method relevant unless we're comparing it to other triangles? Is that number that we get when we use the ratio method, is that even a relevant number unless we're comparing it with other slopes?

Loi: Dr. Simon, like I said before, I can clear this whole thing up right here. The ratio method works, because it's like a rate of 3 to 1, so as you're going down, for every 3 you go down, as you're going down, you're also going over 1. . . . So when you said, "I went down a mountain of 3," so I went down 3 and over 1, down 3, over 1, and that will eventually lead you to this point and that's why you have that line. Know what I mean? . . . and that's your rate and that's what the down grade is. So it's not like an arbitrary number.

Although she uses the mathematical term "rate", Lois tries to adjust her explanation for communication with this particular group. She attempts to explain why the ratio method works and to illustrate what the slope number means by building off the explanations offered by others.

Summary and Discussion

This discussion is remarkable for both its content and the roles played by teacher and students. For more than 30 minutes of this class and two previous classes, the talk centers around a single problem: decide whether or not the ratio and/or difference methods will allow one to order ski slope ramps by steepness from only the measurements of their height and length of base. The students decide first that the ratio method will accomplish this goal through analysis of self-constructed drawings, and then continue the discussion, building off each other's contributions toward increasingly convincing justifications of these ideas. They also decide that the difference method will not provide a valid description of slope because the numerical results do not reflect what they see in their drawings. The students spontaneously generate and discuss questions about the meaning of the mathematical objects and relationships they are studying:

Eve asks what is the meaning of the numerical value of slope; Lilly asks what a 7% grade might mean; and Toni asks if the ratio is meaningful in referring to one slope as opposed to comparing two or more. The students make connections in this discussion between each other's ideas, and between the mathematical ideas and their outside experiences.

This interaction provides evidence that the teacher and students have created a mathematics community in which the students' job is to probe mathematical ideas generated in the process of solving problems. To participate competently in this class, they must make these ideas personally meaningful and prepare themselves to articulate those ideas in discussion. Students view themselves as responsible for listening to and trying to understand each other's ideas, monitoring their own understanding, asking questions of each other if they do not understand, validating each other's mathematical arguments, and trying to relate the different explanations offered. All of this seems to occur in pursuit of the fullest understanding possible for the group, not just for one's self, suggesting a notion that ideas are built as a community rather than as individuals.

The teacher's role in this class is complex. During discussion, it involves monitoring participants' understandings of each contribution, assisting the students' learning by occasionally asking clarifying questions and calling for paraphrases, and supporting the students in explaining their points of view. Simon does not evaluate responses for correctness, but does make judgments about their significance in the context of the on-going discussion, at times asking for responses to a contribution and at other times allowing comments to pass without comment. Simon also heeds the normative practices of the group, consciously highlighting appropriate behavior (e.g., discussion of dissenting points of view, asking questions like Eve's), and explicitly telling students how to behave (e.g., directing their questions to the owner of the idea, reminding students to listen to each other's contributions).

The excerpts from Class 15 demonstrate the extent to which classroom norms and practices were developed which differ greatly from those of traditional school mathematics. The brevity of this report does not allow elaboration of how these norms and practices were established. We can however, point at a few landmarks by which this community passed. Initial class discussions were characterized by discrepant perspectives (students' versus teacher's) on the classroom mathematics activity (see Simon & Blume, in press) and intermittent resistance from some of the students (see Simon & Brobeck, 1993). As the course proceeded, students began to take responsibility for solving novel problems and to identify and anticipate particular demands of the teacher. For example, upon agreeing on some aspect of a problem's solution in small groups, one student would say to another, "You know he is going to ask us to explain why." The data presented here suggests that eventually students went beyond complying with the demands of the teacher to self-directed inquiry and exploration of mathematical ideas. The full paper will explore how this shift came about.

Whereas such a shift would seem to be beneficial to mathematics students at all levels, we believe that it is particularly critical for prospective teachers. It has the potential to alter their relationship to mathematics and to mathematical activity in classrooms. While such experiences are not sufficient for teachers to be able to create such opportunities for their students, they are likely to be necessary.

Footnotes

[1] These prospective elementary teachers will be referred to henceforth as "students" and the teacher educator will be referred to as "the teacher" to indicate the roles that each played in the mathematics course under study.

[2] This material is based upon work supported by the National Science Foundation under Grant No. TPE-9050032. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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TEACHING IN AN ERA OF REFORM: MATHEMATICS IN ELEMENTARY CLASSROOMS

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This research reveals and documents teachers' struggles to teach mathematics in the midst of the current wave of reform. Through a comprehensive attempt to understand elementary teachers, the study discusses social and ideological factors that shape teachers' efforts to change mathematics instruction. These factors are important for our understanding of mathematics classrooms because they influence what is taught as well as how it is taught.

The history of reform in American mathematics education is almost as old as the history of mathematics education itself. Although one can argue that little reform has occurred in the teaching of mathematics since the turn of the 20th century, reform rhetoric has been a constant component of the field's discourse.

Analyses of educational reforms, however, have traditionally focused on quantitative measures of effects, most often with reference to students' achievement scores. We know very little about what teachers actually do in their classrooms (Cuban, 1993) and our comprehension of efforts to change usually does not go inside the classrooms (Cohen & Ball, 1990).

The research reported in this paper is interpretive in nature. It is an attempt to understand elementary school teachers and their struggles to teach mathematics in the midst of the current wave of reform. Through in-depth case studies based on detailed classroom observations, and focussing on the

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Curriculum and Evaluation Standards for School Mathematics (Standards), published in 1989 by the National Council of Teachers of Mathematics (NCTM), this paper reveals and documents how teachers adapt reform documents in their attempt to adopt them. More specifically, this research sheds some light on the roles ideological and social issues play in the practice of teachers as they strive to offer their students what they see as appropriate mathematics teaching.

The Project

The research project consisted of intensive observations of three female elementary teachers—one fourth-grade and two third-grade teachers— from different schools in different school systems. With varying degrees of awareness, these teachers knew about the Standards before the observations began, and they wanted to align their teaching with their interpretations of this document. These teachers, however, were not involved in major research and/or developmental projects. They were classroom teachers, typically left alone to interpret and implement recommendations for change in school mathematics.

The three schools in the project were in three small, mid-western towns, and all children in the classrooms observed were white—which was basically true for the schools' populations as well. The 3 classes had about 20 students, although the schools' enrollments ranged from 300 to 1,000 children. The schools varied in the socio-economic background of the children, measured by the percentage of free or reduced lunches served, and in parents' educational credentials and types of job.

Two of the three teachers were observed between September 1993 and January 1994. The third teacher joined the project later, substituting an initial participant, and was observed between December 1993 and April 1994. During these periods of time, Sztajn was in each classroom for a whole week approximately once a month , completing four weeks of observations of each teacher. The observations were not restricted to mathematics classes. Rather, during each week at the schools, the researcher was in the classroom every day, for most of the day, experiencing the routine of the teachers and their students in the three schools.

While in the classrooms, Sztajn assisted the teachers with traditional classroom work such as grading papers, passing out materials, and helping students complete their assignments. However, in an attempt to understand the teachers' visions of mathematics teaching and learning, the researcher was more often an observer of than a participant in the classes. Neither researcher was involved in the planning of any school or classroom activities.

After each week in a classroom, Sztajn and the teacher met for hour-long, audio-recorded, unstructured interviews about the class and the mathematics taught. Teachers had access to transcriptions of all interviews and could further reflect and elaborate on the issues discussed. At the schools, Sztajn also interviewed other teachers, the principals, and some parents of students. These interviews offered some information on the social context in which the teachers work, and illuminated some constraints and expectations that surround their work.

Sztajn and Lester met weekly to discuss possible themes emerging from the data. Initial trends considered during the field phase of the project served as conversation topics for the interviews with the teachers, allowing the researchers to gather further evidence on issues of interest. For each teacher, all transcriptions and classroom fieldnotes were coded (Miles & Huberman, 1994), and separated into seven main categories: the teacher, her students, her class, the school, parents, general educational issues, and her mathematics teaching. These categories were used to describe the teachers and to support the researchers' claims about each teacher's vision of education, in general, and mathematics education, in particular.

The Case of One of the Teachers

This report focuses on one of the cases studied in the project. It briefly portrays one teacher and her practice, presenting a few elements of her ideological (i.e., value-rich) vision of the world (Ernest, 1991). The teacher is situated within the social milieu in which she is embedded. Due to space limitations for this paper, we do not present here the evidence used as warrants for our claims and conclusions about the teacher's vision. A pseudonym is used for the teacher.

Teresa M. Walker

Teresa is a smiling, talkative person who rarely raises her voice. She is in her early forties, has shining green eyes, and brown hair with bangs. Teresa was born and raised in a very small town in the midwest where her parents were factory workers. She attended a Catholic school from grades 1-8 and went to the public high school in her town. Teresa graduated in 1974, with a degree in elementary education from a medium-size state university, and substitute taught for three years. From 1977 to 1984, Teresa had a non-teaching job outside education. In 1985, when her only son began kindergarten, Teresa decided to start teaching again. Since then, she has taught each of the elementary grades, 1 through 5. This is the second year Teresa is teaching third grade.

For the past three years Teresa has been teaching in the K-5 school where she now teaches. About 40% of the children at the school qualify for free or reduced lunch, and the school is the only one in its system that has a breakfast program. Over 50% of the students live with single parents or have gone through at least one divorce situation. The principal explained that most of the parents hold manual labor jobs and many of them are not high school graduates.

Teachers assume that most of the parents they deal with do not like school, and are not interested in their children's education. It is also generally agreed among teachers that most children they work with are not "ready to learn." These students, teachers say, come from "poor" environments without having the "correct" prior-to-schooling experiences. Therefore, they do not bring to school the "appropriate" knowledge required to succeed. As Teresa puts it, her students are linguistically and mathematically deprived. Besides, she notes, they are not "ready to learn" because they do not have the materials they need in class, and they often come to school hungry and sleepy.

Teresa also claims her students are not "emotionally ready" for the social interactions that take place at school: they lack social skills and bring "faulty" behaviors with them. Teresa thinks it is teachers' role to teach at school what parents "fail" to teach at home. Therefore, she feels that teachers have more work to do in low socio-economic schools. Teresa regrets that extra-curricular activities, not directly related to content, take a lot of her class time.

This school has traditionally had a high failure rate. Now, it is trying to move towards portfolio assessment and parent-teacher conferences. Teachers are also evaluating students according to their individual progress, in what they call a non-failure program. Teresa attributes failure to students' low ability, "bad" attitudes towards education, and "improper" home atmosphere. She likes the idea of not comparing her students, but she says children need to know when they are not achieving as expected. Children also need to learn at school how to deal with failure, because one frequently fails in life. For her students' portfolios, Teresa includes all chapter tests and the work-sheets students had difficulties with, in order to show parents what their children are not learning.

Teresa coaches during the summers and sports is her metaphor for teaching. In the classroom, her students need to be organized, in their places, ready to pay attention, and follow her directions. They need well-defined tasks, clear instructions, guidance, and supervision. They also need a lot of drill and practice to master skills. Coaching her students, Teresa does most of the talking in her classes. She presents the topics, asks questions, and expects the children to work hard until they become proficient. She tells her students they need to exercise their brains like muscles in order to learn.

Teresa does not consider herself a "math type." Mathematics was never easy for her, and she only learned it through hard work. Mathematics for Teresa is a set of rules to be used and remembered, and school mathematics is a list of topics to be taught. The Standards represents the most current version of this list, and because Teresa thinks she teaches what is in the list , she assumes she is following the NCTM recommendations. Teresa also uses a textbook "based on" the Standards to plan and present material to her classes.

For Teresa, learning "topics" like reasoning and problem solving is as important as memorizing basic facts. However, Teresa claims she has two main difficulties in teaching higher-order thinking skills. One, she was never taught them and she tends to teach the way she learned—it is hard to change, she says. Two, it is hard to control classroom discipline when many students cannot follow her instruction.

Because time is a continuous constraint for Teresa, due to all the "extras" she has to teach her students, and because Teresa has many low-ability students (which in her mind is a result of the type of community they come from), Teresa prioritizes her mathematics teaching: first, discipline; second, skills; and third, whenever possible, thinking. Therefore, Teresa's vision of her students, their learning abilities, their family background, and their educational needs, among other factors, shape her mathematics instruction.

Discussion

To understand Teresa's mathematics teaching, one needs to paint a holistic picture. Teresa's vision of children, of her students in particular, her ideas about schooling, about society, and her theories of learning, cannot be disconnected and examined separately. Her ideas about the nature of mathematics and mathematics teaching and learning also need to be considered.

This comprehensive attempt to understand one elementary teacher reveals the presence of ideology in mathematics teaching. As value-rich, systems of beliefs, ideologies shape teachers' efforts to change mathematics instruction, influencing what is taught as well as how it is taught. The social context of teaching both constrains and facilitates changes.

To date, far too little research has investigated the roles played by ideological and social issues in mathematics teachers' practice. This unfortunate state of affairs must change if we are to have any hope of developing an understanding of how teachers make sense of the mathematics reform rhetoric.

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CHANGING PRACTICE: DESCRIBING MATHEMATICS TEACHERS' DEVELOPMENT THROUGH A MODIFICATION OF PERRY'S SCHEME

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A modification of a scheme proposed by William Perry (1970) is described in an attempt to illustrate how mathematics teachers go about changing their practice to incorporate current reform recommendations. This partially hierarchical framework evolved while attempting to characterize the two-year experiences of four middle school mathematics teachers. Data from one case study is used to elaborate the framework.

One issue central to the mathematics education reform movement is the role of authority in teaching and learning. Expecting students to understand and explore mathematical ideas requires them to accept much of the responsibility for determining, for example, appropriate procedures and methods to solve problems. This contrasts with the teacher-centered (and often dominated) dialogue that takes place in many mathematics classrooms. Perry's scheme (Perry, 1970; Copes, 1982) emphasizes this issue of authority.

Perry's scheme has been used to describe the experiences and conceptions of preservice and beginning mathematics teachers (see e.g., Cooney, 1993). We believe that Perry's scheme can also be usefully adapted as a tool for analyzing the struggle experienced mathematics teachers may undergo as they strive to reform their own practice. However, we suggest that there is an implicit hierarchy in the original Perry framework which, while fully appropriate for the purposes to which he put it, is less useful to describe what we have observed with mathematics teachers. Thus we suggest four categories which parallel, but do not strictly duplicate, four of Perry's main divisions.

In so doing, we are aware that Perry built his model as a result of research done with male undergraduate students at Harvard during the 1950's. Thus, any framework which builds on Perry's is subject to a variety of criticisms: most centrally, mathematics teachers in the 1990's are not all males who attended Harvard four decades ago. We also recognize that other schemes (e.g., Belenky, et. al., 1986) have covered ground similar to that which Perry examined, using considerably different perspectives. Nevertheless, we believe that the framework we propose is useful for exploring the struggle for pedagogical self-reform and that its applications may not be limited to the particular individual to whom we apply it in this paper.

Furthermore, we do not claim to be making use of Perry's original categories in exactly the same sense that he employed them. Perry was examining the intellectual and moral development of a relatively young group of students. We are interested in looking at the development of mathematics teachers who are struggling to reform their practice. This group clearly consists of a far wider range of individuals in age, ethnicity, gender, and socio-economic background than the pool of students Perry studied. The examples cited in this paper are taken from our study of a veteran teacher, one who is undergoing a process which differs in many respects from that of an undergraduate student. Yet, there are sufficient parallels to make our adaptation of Perry's terminology heuristic. We are indebted to Perry, but responsibility for any criticism this modification of his scheme may engender is ours alone.

To both test and illustrate the usefulness of this scheme as a tool for examining important aspects of mathematics education, we draw on a study currently in its second year that we have been conducting of Mr. Burt, a veteran sixth-grade teacher .[1] We suggest four broad categories: dualism, multiplism, relativism, and pragmatism.

Dualism

We will term dualistic any set of beliefs and practices that tends to indicate that the holder views the world as comprised of absolute right and wrong alternatives. Every situation, decision, action, thought or belief can be given one of two mutually exclusive and opposite ratings. Individuals who operate dualistically may view these as correct/incorrect, right/wrong, plus/minus, good/bad, good/evil, or through other binary constructs. Crucial to dualistic thinking, however, is the predominant belief that there are no meaningful shades between the two extremes. Also typical of dualistic thinking is the belief in an absolute authority which lies outside the individual. This authority is the ultimate arbiter that decides which pole properly describes a given phenomenon. For mathematics teachers who operate dualistically, this authority may be the teacher when evaluating student work, as well as the textbook, and/or the community of experts, which may include professional mathematicians, teacher educators, researchers, district supervisors, curriculum coordinators, department chairpersons, etc.

The following episode (see Wilson & Goldenberg, 1993), from a lesson on fractions that Mr. Burt taught in November 1992, is in many ways representative of the dualistic nature of his overall approach. Mr. Burt began this particular lesson by modeling (with student input) on the overhead projector the solution to the first exercise in the lesson. He then asked students to complete the next few exercises. Mr. Burt then modeled solutions to the assigned exercises. For example, on the overhead he completed the exercise shown on the left of Figure 1 by drawing a vertical altitude to the middle triangle. Rosalind, a student, asked if she could show a different way to do the problem. Mr. Burt invited Rosalind to the front of the class and she sketched a solution like the one shown in the right part of Figure 1.

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[1] For a description of Mr. Burt, the research setting, methodology, and an account of our first year's experience with him, see Wilson & Goldenberg, (1993).

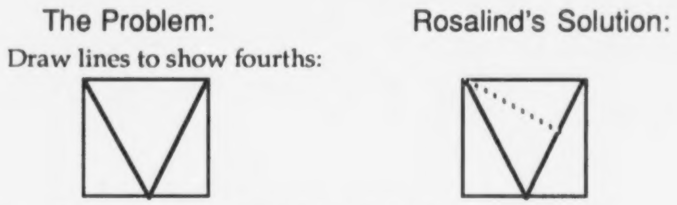

Figure 1. Fraction problem and one student's solution.

Upon seeing Rosalind's solution, Mr. Burt responded, "I don't know if that would be right or not. I'd have to really look at it. I can't tell, I'd have to measure it." He then measured the lengths of the two segments along the base of the triangle she had divided, and announced, "Nope, that won't work because this is longer this way that it is this way. But that was a good idea, you were thinking. Thank you Rosalind."

One thing illustrated by this episode is the extreme directiveness of Mr. Burt's instruction. Almost never during the first year did we observe him attempt to involve students in deciding the appropriateness of strategies or solutions; he was the sole arbiter of correctness. Nothing we saw in Mr. Burt's mathematics teaching would challenge the notion that he believed there was a single correct answer and a single best way to arrive at it.

Multiplism

Multiplistic thinking is characterized by an increasing awareness of multiple perspectives which is overlaid on a primarily dualistic framework. Multiplists recognize that there are various opinions on any given subject, more than one way to view phenomena, and different approaches to dealing with the world. However, multiplists generally believe that there is always one best or correct choice among these possibilities. Unlike the strict dualist, multiplists may acknowledge that there is a spectrum along which varying viewpoints can be distributed. However, they share with dualists the belief that "objectively" there must be one best opinion, idea, belief, method, approach, or answer. Again, multiplists tend to share with dualists the sense that there is some outside authority that can correctly decide which among many choices is best. For a multiplistic teacher of mathematics, this may mean that while there are many possible approaches to problem-solving, there is always one best way to do a given problem. Multiplists may tolerate the idea of diversity, but they strive for perfection, seeking for what in another context Hofstadter (1985) has termed "God's algorithm."

We observed a number of examples of multiplistic thinking during our second year of working with Mr. Burt. For instance, in reference to the use of manipulatives, he claimed that although there are many effective ways to incorporate them, there is generally one best or most appropriate technique; the reason he tends not to use manipulatives much is that he is not familiar with these techniques:

> I want to be able to use those [manipulatives], but right now I am not facile enough to use those and whip those out without spending time, and I want to do more of that this summer. I have to just somehow make time to start doing that. It's nice to have a group of about five students to try that out during the summer. They could meet once or twice a week and experiment with the stuff to see if it works. Better than experimenting on 30 kids.

For Mr. Burt, having his students use manipulatives unless he is sure what he is doing with them constitutes "experimenting" in a negative sense. That the students might benefit from their own experiments with manipulatives does not seem to be part of his reasoning. If Mr. Burt feels insecure about something, he is not likely to try it in his classroom. The effect of insecurity on his pedagogy is consistent with a theme we witnessed throughout our observations of and conversations with Mr. Burt: he has a strong need to find the best way to do and understand mathematics.

Relativism

Relativistic thinking can be described as multiplism shifted away from dualism. The relativist shares the multiplist's belief in many viewpoints, but unlike the multiplist maintains that no perspective is privileged. For the relativist, all points of view have an equal claim to validity. Authority no longer lies outside the individual. Each person is equally correct in claiming to be right; however, each person is equally wrong in making such a claim. It can be suggested that for the strict relativist, there is no authority. While such a position may seem an unlikely one for a mathematics teacher to hold about mathematics, it seems plausible that teachers could hold a relativistic view of mathematics teaching. That is, a relativistic mathematics teacher may believe that there is no approach to mathematics pedagogy which has a claim of superiority over another. While the relativist may be open to a wide variety of innovations, it may be difficult, if not impossible, for her to effectively choose among the many available options.

We find little if any evidence in Mr. Burt's practice to illustrate relativistic thinking or practice, although we have seen it in observations of other teachers. We do not find this absence surprising in Mr. Burt's case: we would not expect to see him thinking relativistically because he is too much of a realist, too grounded in a faith about his world, to believe that there are no right answers to the questions he considers.

Pragmatism

This position can best be described as post-relativism, and is referred to in Perry's scheme as commitment. The pragmatist recognizes, along with the multiplist and relativist, that there are available many viewpoints, methods, ideas and approaches. Unlike multiplists, pragmatists do not resent confronting many available choices. Unlike relativists, they do not despair at having to make meaningful selections from the wide array of possibilities. Unlike dualists, they need not and generally will not claim that there is a single best choice. Pragmatists instead believe that while there are many good ways to think and act, there are many choices that are ineffective, as well as those that deserve to be shunned outright. Furthermore, they believe it possible to create at least a partial hierarchy among available options. While pragmatism entails the belief that some choices are better than others, the authority for so choosing is internally constructed. Each individual weighs the possibilities available and makes considered decisions among them. At the same time, pragmatists remain open to the possibility of change, not because of an inability to choose, but because they believe there is no ultimate outside authority against which to measure things. Pragmatists remain committed to a position, viewpoint, or method only insofar as it remains effective and useful. At the same time, they remain aware of the contingent nature of any belief or belief system (Rorty, 1989).

For mathematics teachers, operating pragmatically means being open to a wide variety of pedagogical approaches that can and will be evaluated as to their efficacy by the individual practitioner. This evaluation is shaped by the individual's personal beliefs and experiences (i.e., teachers are likely to teach using methods they understand and are comfortable with, as well as those that conform with their personal beliefs about learning), but is also modified by an awareness of alternative learning styles and needs of students. Thus, a mathematics teacher who is primarily shaping her practice through pragmatism is likely to be eclectic in a way that differs significantly from the relativist. While the relativist often suffers from a kind of intellectual and moral paralysis, unable to make meaningful choices from the host of possibilities due to an underlying belief that essentially all positions are equally valid, the pragmatic teacher chooses what seems to work. Since it is unlikely that any one approach or explanation will suffice for all students, the pragmatic mathematics teacher models a range of solutions and approaches to problems; she encourages students to think about mathematics from varying perspectives while still being critical about what constitutes adequate, efficient and elegant mathematical argument and thought. Perhaps not coincidentally, much of what we describe here is consistent with many of the points raised by the *Standards.* (NCTM, 1989).

Analysis of an event that occurred during a lesson on October 7, 1993 contrasts strongly with the Rosalind example and illustrates Mr. Burt operating in a more pragmatic style. In one of the exercises, students were asked to draw lines in a picture of a rectangle to illustrate fourths. The diagrams shown in Figure 2 illustrate two solutions suggested by students. Mr. Burt posed the question, "would the area of the two [shaded] boxes be the same?" Most students thought the correct answer was "yes," although several did not agree. Mr. Burt had one student explain why she thought the two boxes had the same area, after which he asked one of the dissenting students to explain his reasoning. This discussion started a debate among students, which neither Mr. Burt nor his students settled during that class. Instead, he asked students to think about the situation and write up their explanations for homework.

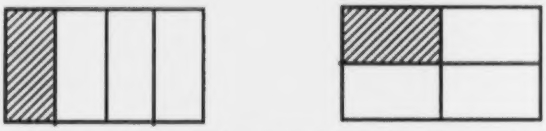

Figure 2. Alternative solutions to a student exercise.

We must note that in an interview following this lesson, Mr. Burt's response to questions about this incident suggest that he had not followed up on the homework assignment. He indicated that some students came back to him having determined that the areas were the same. But our questions about this incident seemed to catch him very much by surprise; based on our numerous interviews and informal discussions with him over two years, such an instance was unique. Further, while he claimed that there had been similar situations during the year in which he had students debating or discussing mathematics in class, he could not recall any specific incidents. While we cannot conclude unequivocally that no such incidents occurred, these data imply that such activities were not salient for Mr. Burt. Thus, we do not suggest that Mr. Burt has either made a permanent and dramatic change in his thinking or pedagogy. Rather, we offer this example as evidence of his capacity for pragmatic thinking and teaching.

One crucial implication here is that Mr. Burt is caught among numerous competing forces: his general belief that there is a right way to teach, a right way to learn, and specifically a right way to do mathematics; his realization that his own experience as a learner of mathematics and of mathematics pedagogy is lacking; his confidence in his specific understanding of important mathematical ideas; his insecurity regarding the "correct" use of manipulatives; his recognition of the need and sincere desire for innovative mathematics teaching; and twenty-one years of ingrained, successful teaching practice.

Concluding Remarks

Those who are familiar with Perry's original scheme will recognize that we have taken major liberties in the above definitions. We are employing only four categories: there is no attempt here to create the large number of categories Perry and his colleagues used to map the various transition phases they observed in the development of the students they studied. Our use of the terms 'multiplism' and 'relativism' departs significantly from the definitions Perry developed. There are several important explanatory points we wish to make in this regard. To begin, we are creating categories that developed as an interplay between our reading of Perry and other writers on the one hand, and individuals we have been investigating for two years on the other. These four broad categories represent not points on a continuum through which we expect mathematics teachers to move on their way to pedagogical perfection, but rather territories which any given practitioner may visit in a given day of teaching. That is, we suggest that the story we have to tell is not one of linear progress toward an ultimate goal so much as it is one of changing emphasis as a teacher becomes more comfortable in previously avoided spaces. The result is not an individual's dramatic transformation from rigid, dualistic practice to flexible, pragmatic practice. It is instead a less operatic, but admirable human story of a successful teacher's struggle to become more effective because of his belief that his students are worth the effort.

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TEACHER PARADIGM: FORM AND TRANSITION

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This paper consists of a critical evaluation of one teacher's dynamic transition between two belief systems and is intended to give insight into teacher paradigm shifts, changes which must occur for continued success in mathematics education reform. Identification of the source of tentative first steps towards paradigm shifts and the problems or rewards encountered in maintaining each element of change are specifically examined.

Lampert (1985) suggests that she and other teachers manage dilemmas "by putting the problems that lead to it further into the background and by bringing other parts of my job further to the foreground" (p.186). This view of teaching as managing dilemmas, rather than making didactic choices, presents decisions in a light of "coping rather than solving" (p. 189) and provides a framework for the analysis of 32 hours of observation and interview data gathered over a six-month time frame with Robert Stevens, the project's focus teacher. Additionally, Lampert's description of seeing "the person she wanted to be" (p. 188) and how this vision assisted her in managing dilemmas provides a framework to analyze this teacher's voice as he talks about "seeing myself as I teach using my constructivist philosophy." Lampert's view of the "contradicted 'self' as a tool of her trade" also allows insight into Robert's story of meeting and managing conflicts (p. 190).

Although Robert expressed comfort, satisfaction, and pride with his newly adopted constructivist philosophy, he spent more time discussing his frustration with his desire for a "theoretically pure constructivist classroom" and his inability to achieve that desire. He also expressed anxiety about "all those years, I thought I had done things right. Does what I know now mean that I shortchanged all those kids in all those classes?" Although Robert had taught for 25 years, he continually talked about his mentor teacher, Mr. Stellar. "How is it that I could have learned so much mathematics from Mr. Stellar if Mr. Stellar had taught things in the wrong way?" "If Mr. Steller could see me teaching now, would he approve?"

Further research in this area may confirm that instability and discomfort in transitions are commonplace for many teachers. Recognition and use of the *conflicted self as a tool* might assist mathematics teachers, like Robert, in accepting the teaching dilemma they face as they balance pragmatic classroom needs with their desire to adopt a particular teaching philosophy. Additionally, Robert's genuine concern in disappointing his mentor may imply that mentor teachers might be well-served to continually remind student-teachers that although they appear to have all the answers, they too manage dilemmas.

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EXTENDING K-8 TEACHER DEVELOPMENT IN MATHEMATICS TO THE SECONDARY LEVEL: A MODEL FOR DISTRICT SYSTEMIC REFORM

Emily Dann, Roberta Schorr, Alice Alston

Rutgers University

Most reform in school mathematics has focused on teaching and learning in grades K-8. While it is recognized that change in the teaching and learning of mathematics needs to be extended to the secondary level, relatively little attention has been given to teacher development in the high school that considers the complexity of systemic reform. This Poster Session will describe a model that is the outcome of the New Jersey Systemic Initiative in Mathematics, Science, and Technology and which takes into account transition from work with K-8 schools to work with the receiving high schools. Emphasis will be on working in partnership with administrators, Boards of Education and teachers in environments which support both teachers and students thinking. The following components will be addressed: long term in-service teacher development through modeling and team teaching; classroom situations using formats that consider a range of learning styles; in-school sessions for teacher planning, sharing, and discussion among colleagues; curriculum organization that offers support for the integration of mathematics with technology and other subject areas; implementation of more conceptually based forms of assessment and facilitation of parent group activities which foster community understanding of changes taking place in mathematics education.

Mississippi Mathematics Improvement in Secondary Schools Project (MISS)

Dr. Barbara J. Dougherty and Dr. J. Larry Tyler

University of Mississippi

As teachers incorporate changes in their secondary classrooms, three factors, content, instruction, and assessment/evaluation, must be orchestrated to provide consistency in expectations of students and to promote the empowerment of students as mathematicians. Project MISS has focused on staff development techniques to promote consistent and cohesive changes in those three areas of secondary mathematics instruction. Teachers participate in activities that are designed to use **their** philosophical and practical beliefs and experiences to bridge nontraditional content, instruction, and evaluation with their current practices.

These activities have been aligned with Vygotsky's premises about learning as evidenced from research in the Hawaii Algebra Learning Project. Specifically, learning is enhanced when 1) concepts are introduced through problem-solving tasks; 2) social interaction includes speaking, writing, listening, reading, and modeling; 3) connections are made with other mathematical concepts or with other disciplines or experiences; 4) concepts are moved to skills over a period of three to eight days; and 5) all students are challenged.

The focus on student learning provides the impetus for any classroom changes and, thus, a solid underpinning for teacher decisions about content, instruction, or assessment. This cohesive approach, based on student learning, is a systemic model for change in mathematics education.

A REFLECTIVE CYCLE: THE EVOLUTION OF A MODEL OF TEACHER CHANGE

Thomas G. Edwards

The Ohio State University

The process of teacher change is complex and remains somewhat intractable. In order to better understand the process, the development of a model is a logical starting point.

A case study methodology was used to study three teachers who were implementing an innovative mathematics curriculum for the first time. Evidence of changes in their instructional practices was sought. Because making changes in one's instructional practice is often problematic, these teachers were viewed as active learners in a problem solving activity as they made changes in practice to accommodate the new materials they were using. From this perspective, constructivist theories of learning provide a means of analyzing the process of teacher change.

The construction of knowledge has been described as a cyclic process. The key points in the cycle are *interaction* with one's environment leading to some form of mental *dissonance*, which results in a cognitive *restructuring*. This completes a cycle, because interaction with one's environment is inevitable. In such a model, the individual's reflective activity is the glue which holds the process together. When these ideas are used to study the process of teacher change, mental dissonance translates into some aspect of practice that becomes problematic for the teacher, and restructuring takes the form of change in teaching practice.

These constructivist concepts by themselves, however, do not adequately account for the process of change in the teachers who were studied. Two of them changed in very similar ways, while the third changed in ways that were quite different from the others. In order to explain this difference and to more fully describe the process of change, the teacher's underlying belief structures, particularly their beliefs about the nature of mathematics itself, must be considered. Further complicating any model of teacher change is the uniquely human ability to reflect upon one's own thoughts. Indeed, these teachers' reflective activity was central to the changes in practice that were observed in each of their practices.

Thus, the model of teacher change presented will weave the constructivist notion of a reflective cycle with teachers' beliefs and metacognitive activity.

TEACHER INQUIRY GROUPS:
COLLABORATIVE EXPLORATIONS OF CHANGING PRACTICE

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Current calls for change in the way mathematics is learned and taught require teachers to develop 1) a broader and deeper view of mathematics; 2) new beliefs about how students learn mathematics; and 3) new teaching practices consistent with these new beliefs. Such shifts in knowledge and belief require focused attention over extensive periods of time on many subtle and specific issues linked to classroom practice.

The Mathematics for Tomorrow (MFT) project is developing and examining the effects of a new type of social support for change in teaching. We have created an ongoing discourse community that we call "Inquiry Groups" in which teachers meet regularly to critically examine their own and each others' mathematical and pedagogical knowledge, beliefs, and practice as they work to make changes therein. This poster will explore some of the questions we're currently pondering about the nature and effects of these communities: What experiences, beliefs, skills, and social-cultural supports help teachers become "critical friends" in investigating each others' teaching practice? What issues—mathematical, pedagogical, epistemological, emotional, political, and so forth—arise for teachers in these groups? How do teachers make this new form of discourse and investigation their own?

Negotiating A Technological Shift: Teacher Perception of The Implementation of Graphing Calculators In The Classroom

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The purpose of this research is to describe the perceptions of teachers as they engaged in instruction using graphing calculators for the first time. Data collected from 261 teachers who participated in the five month Virginia Network for Technology (VANT) Outreach interactive telecourse, Integrating Technology Into Mathematics and Science Instruction, included pre- and post-questionnaires, student course work, and journals describing initial instructional experiences using graphing calculators. An analysis of data from questionnaires shows that teachers who hold a more rule-based viewpoint about learning mathematics were more likely to view the use of graphing calculators as inhibiting student understanding. Whereas, participants who were less rule-based were more willing to adopt the use of calculators as a part of instruction. Overall, participant views changed significantly ($p < .001$ level) in favor of viewing the use of the graphing calculator as a "thinking tool" to enhance conceptual understanding. Participant journals were coded and analyzed to gain in-depth understanding of teacher beliefs, thinking, and behavior. The emerging themes and patterns revealed that rule-based and non-rule-based teachers negotiate the implementation of graphing technology using a combination of procedural and conceptual approaches to instruction. Rule-based as compared to non-rule-based teachers are more likely to take on the role of lecturer than facilitator, to use individualized instruction, and to judge success of an instructional activity based on affective reactions of students rather than student understanding.

Analysis of instructional strategies used by participants shows that teachers were more likely to restructure their lessons towards inquiry and conceptual teaching as they gained confidence in the use of graphing calculators and as they reflected on the benefits of inquiry-oriented approaches to learning which they experienced during the VANT Outreach course. Implications for integrating technology into instruction are discussed in this poster.

Teachers' Conceptions of Mathematics

RESEARCH PAPERS

Teachers' Graphical Representations Of Rate Of Change 261
Sarah Berenson & Glenda Carter

Distributive Flaws: Latent Conceptual Gaps In Preservice Teachers' Understanding Of The Property Relating Multiplication To Addition 268
Stephen Campbell and Rina Zazkis

Elementary Teachers' Understanding And Implementation Of Unitizing Operations 275
Tena Long Golding & Merlyn Behr

Elementary School Teachers' Perceptions Of Algebra: The Role Of Modelling And Techn. 282
Barbara J. Pence

Learning Mathematics While Teaching 289
Susan Jo Russell, Deborah Schifter, Virginia Bastable, Lisa Yaffee, J. Lester, & S. Cohen

Teachers' Changing Conceptions Of The Nature Of Mathematics: A Developmental Model 296
Deborah Schifter

The Fundamental Theorem Of Arithmetic: Used and Confused 303
Rina Zazkis and Stephen Campbell

ORAL REPORTS

Calculus Teachers' Recognition Of The Concept Of Derivative 310
Rodrigo Cambray-Nunez

An Investigation of Preservice Teachers' Understanding of Rational Numbers 311
Christine Carrino Gorowara

Prospective Secondary Mathematics Teachers' Subject Matter Knowledge: The Concept of Limit As An Example 312
Brenda Lee

Hands-on Assessment of Elementary Teachers' Understanding of Whole Number, Fraction, and Decimal Concepts, Principles, and Processes 313
Olga M. Ramirez, John E. Bernard, and Arturo Olivarez

POSTERS

Preservice Teachers' Perceptions Of Procedural And Conceptual Problems 314
Kathryn Skipper Reinke

TEACHERS' GRAPHICAL REPRESENTATIONS OF rate OF CHANGE

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A number of researchers have examined students' ideas concerning the interpretation and representation of graphs, but few have examined teachers' graphical ideas. This initial investigation examined how teachers graphed or represented a real-world, rate of change problem. Seventy teachers were asked to consider the "John and Mary" problem taken from the NCTM Standards *by graphically representing the events. Fifteen teachers represented the problem as one of linear relationships while controlling for two variables with appropriate slopes, intervals, and intersections. Twenty-one teachers were not able to consistently control for two variables and nearly half of the teachers perceived walking or running as a non-linear relationship.*

BACKGROUND

Fennema and Franke (1992) stated that most scholars agree to the importance of a teacher's mathematical knowledge. However, there is some evidence that preservice and inservice teachers, particularly at the elementary and middle grade levels, do not have deep mathematical understanding (Brown, Cooney, & Jones, 1990; Post et al., 1991). Indeed it is not surprising that teachers who learned mathematics at the nadir of algorithmic learning may not have the complex cognitive structures of mathematical concepts required to reform the teaching and learning of mathematics.

The NCTM *Standards* (1989) changed the focus of learning mathematics from the memorizing of algorithms to the understanding of concepts, and from theories of behaviorism to theories of constructivism. Theorists such as Piaget, von Glasersfeld, and Vygotsky (Davis, Maher, & Noddings, 1991) have contributed the following constructivist principles: a) learners construct their knowledge from interactions with the world, b) considerable variation may exist between an individual's constructions and the "expert's", and c) an individual brings previously constructed ideas to the classroom. These ideas can serve as a scaffold for new ideas or be used to alter the ideas or conceptions that are not viable.

For this research, we were interested in the graphical ideas of middle grades and high school mathematics teachers. Janvier, Girardon, and Morand (1993) clarified the meaning of representation by dichotomizing the term. External representations belong to the physical world and include charts, graphs, diagrams, written symbols, and manipulatives. Internal representations belong to the intellectual world and cannot be directly observed. Schema, concepts, ideas, and mental objects are examples of internal representations. Teachers' knowledge of external representations was found to be weak by some researchers including Norman (1992) and Orton (1988). These results raised some interesting questions with respect to how well teachers know the representations of the content they teach (Fennema & Franke, 1992).

Bell, Brekke, and Swan (1987) stated that the way graphing is taught in most traditional syllabi does not prepare students for interpreting graphs that communicate everyday information. High school students' graphical misconceptions found by these researchers concerned: a) graph as picture, b) control of two variables, c) types of relationships, d) gradients and intervals, and e) fixations. Clement (1989) found that college students made two types of errors: graph as a picture and misinterpretation of certain graph features. Therefore, the questions that this research posed were: a) How do teachers graph an everyday occurrence involving the rate of change of two variables? b) What alternative conceptions do teachers bring to represent a rate of change graphing problem?

METHOD

The subjects of study were 70 middle grades and high school teachers who were participants in several algebra workshops in Summer '93. They had elected to come to these workshops because they would be teaching algebra for the first time and/or they wanted to learn new methods for teaching algebra. During the workshops, these 70 teachers considered the "John and Mary" problem taken from the Curriculum and Evaluation Standards of the National Council of Teachers of Mathematics [NCTM]:

> *Mary and her brother John leave home together to walk to school. Mary thinks they are going to be late, so she starts out running, then tires, and walks the rest of the way. John starts out walking and starts to run as he nears the school. They arrive at the same time* (NCTM, 1989, p. 100). "Draw a graph that represents Mary's and John's trips to school."

Teachers' responses were analyzed to determine the number of teachers who gave a linear interpretation, controlled for two variables with appropriate slopes, and represented John and Mary starting and ending together. The remaining responses were sorted five additional times to determine different categories of alternative conceptions and the frequency of those conceptions. It was possible to identify several categories of alternative conceptions represented within a single graph. The base of the percents in the frequencies reported is 70 in all cases. Five categories were identified and defined as follows:

Category 1: One variable and discrete data. The graph represented the problem as having only one continuous variable (bar graph) or two discrete variables (points). Graphs showing these alternative conceptions were excluded from further sorts.

Category 2: Controlling for two variables. The slopes of the graphs were examined to identify slopes of 0, undetermined, and negative values. For example, a horizontal line indicated that walking took time but covered no distance or a vertical line that indicated that running covered much distance but took no time.

Category 3: Linear and non-linear relationships. The shapes of the graphs gave evidence that running and walking were perceived as non-linear relationships. (Bell, Brekke, & Swan, 1987).

Category 4: Gradients and intervals. The slopes of the lines or gradients of the curves and their corresponding intervals were examined to detect graphs that were too steep over too little time (running) or too flat over too much time (walking). This is a finer analysis of the alternative conceptions in Category 2, controlling for two variables (Bell, Brekke, & Swan, 1987).

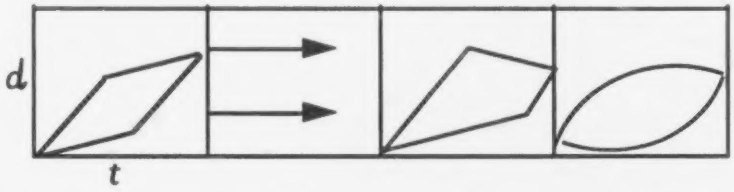
Category 5: Multiple events. In this last sort, the intersections of the graphs were examined to account for two intersections, one at the beginning of the trip to school (0,0) and one at the end (upper right). Note was made of any intersections represented during John and Mary's trip to school.

RESULTS

The first sorting of the data determined that 15 teachers (21%) graphed the problem with linear relationships, controlling for two variables with appropriate slopes, and represented John and Mary starting and ending together (See Figure 1a). The next data sort of the remaining 55 teachers involved the first category of representing only one variable or representing two, discrete variables. Seven teachers (10%) were identified as have these types of alternative conceptions about the problem and were excluded from further sorts (See Figure 1b). In the next sort (Category 2), 21 teachers' graphs (30%) indicated some alternative conceptions about controlling for two variables (See Figure 1c). The sort for Category 3 identified 35 graphs (50%)

that represented running and walking as a curved relationship (See Figure 1d). Also shown in Figure 1d is an example of a Category 4 alternative conception where the gradient and interval are not truly representative of the rate of change in the problem situation. There were 25 graphs (36%) that had this alternative conception. In the final sort, only 8 graphs (11%) had alternative conceptions concerning where John and Mary started together and met.

Figure 1. Different interpretations of an everyday occurrence concerning rate of change.



(a) (b) (c) (d)

DISCUSSION

The number of alternative representations given by teachers to a rate of change problem indicated that additional investigations are needed of teachers' external and internal representations of rate of change. This investigation was limited because there was only one problem and only one translation required in the John and Mary problem: verbal description –> graph (Janvier, 1987). However, it is important to continue to investigate the complexity of teachers' cognitive structures in relation to rate of change and the ease with which they move among different external representations.

Janvier (1987) suggested that a concept, process, or object can have several external representations. Burkhard (1977) described the processes of translating from one representation to another for the concept of variable with four modes of representations: verbal description, table, graph, and formula. These modes of representation and others are also found in the NCTM *Standards* for grades 5-8 (Standard 2 and 10) and grades 9-12 (Standard 6).

Lesh, Post, and Behr (1987) asked the rhetorical question why translations from one representation to another are omitted from mathematics instruction and assessment programs. Answering their own question they observed that translations are difficult to present within a textbook format and researchers are not sure what role translations play in the acquisition or construction of knowledge. The results of this study offer a third explanation, namely, that teachers may not be comfortable with unfamiliar translations.

To further our investigations of teachers' concepts of rate of change we propose that translations can be used to effect a research plan. Janvier (1993) stated that increasing the number of external representations enhanced the individual's internal representation. We conjecture that by increasing the number of translations of a given concept, as researchers, we can increase the quantity and the quality of information obtained concerning teachers' internal representations of rate of change. There are five representational modes in Table 1. Using this table, a research plan is proposed to investigate conceptions embedded within the concept of linear rate of change. The external representations across the top of the table infer a variety of research methods. Written descriptions, graphs, and formulas suggest paper and pencil methods, oral descriptions suggest interview methods and concrete objects suggest performance methods. The empty cells represent other concepts embedded within the concept of linear rate of change. The John and Mary problem has several embedded concepts including slope, control of two variables, and linear functions (See Table 1).

Within the constructivist framework it becomes increasingly important for teachers to be able to translate abstract mathematical ideas to multiple representations for their students. Translations can serve a dual purpose – the first as a teaching tool to enhance students' internal representations and second as a research tool to understand students' internal representations.

Table 1.

Translation Table for Investigating Concepts Embedded within Linear Rate of Change

To From	Written Descriptions	Oral Descriptions	Graphs	Formulas	Concrete Objects
Writ. Des. (John & Mary)	▨		m, (x,y), $f(x)$		
Oral Descrip.		▨			
Graphs			▨		
Formulas				▨	
Concrete Objects					▨

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Distributive Flaws:
Latent conceptual gaps in preservice teachers' understanding of the property relating multiplication to addition

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This study investigates preservice teachers' content knowledge of distributivity with a focus upon explicating lacunas in the conceptually prior notions of multiplication, factoring and equality that contribute to misunderstandings of this important property.

Background and Motivation

Much energy and attention in mathematics education and in research in mathematics education has been focused towards the fundamental operations of addition and multiplication. However, proportionally minuscule attention has been given to the important property relating the two: distributivity. Reports (Weaver, 1973; Blume & Mitchell, 1983) from scarce literature in this area carry a common theme: 'unexpectedly low' and 'dismal performance levels' in (grades 4-8) 'knowledge of and ability to apply the distributive law'. Both studies strongly attack the automatonic assumption that the meaning and significance of this property eventually becomes evident to young minds by 'symbol pushing'. For many learners of mathematics it may never have become evident.

In a previous investigation of preservice teachers' understanding of divisibility it seemed that some of the conceptual difficulties encountered there were more deeply rooted than the study was initially designed to consider (Zazkis & Campbell, 1994). In particular we found evidence to suggest that a significant obstacle to conceptual understanding of divisibility involved a lack of conceptual understanding of the relationship between additive and multiplicative structures. In order to evaluate this conjecture, additional interviews were conducted with the same study group to investigate the nature of their conceptualizations of distributivity in greater detail.

Theory

The constructivist oriented approach we have been using is an action-process-object-schema developmental framework (Dubinsky, 1991). Although ostensibly designed for advanced mathematical thinking, we have found this framework particularly useful in our investigations of preservice teachers' understanding of elementary number theory (Zazkis &

Campbell, 1994). As we approach more 'elementary' concepts of arithmetic, such as distributivity, the data appear to present difficulties that compel theoretical refinement. In particular, in the action-process-object-schema framework the objects initially involved in actions and processes are assumed to be unproblematic. However, we concur with Sfard in noting the 'harmful tangle' of meaninglessness that can result when objects used in higher level processes have not been properly reified (Sfard, 1991).

We believe that the *absence*, or *inadequacy* of conceptual meaning and significance of concepts such as distributivity can lead to the entrenchment of *latent conceptual gaps in understanding* . We must, somehow, be able to theoretically account for the possibility that objects used in actions and processes may not themselves have been properly encapsulated. In this regard, we find it helpful to invoke a perennial distinction and apply it to distinguish between the structural *form* of encapsulation and the meaningful *content* of what is encapsulated. It is important to note that the structural form of an object may be empty or contain meanings that are unintended. Only when the structural form of an object contains 'adequate' or 'appropriate' *conceptual* meaning and significance will we identify that which is encapsulated as the object intended: We have chosen to use the word *lacuna* as a technical term to express when this is not the case. Thus a lacuna can be disguised in the form of an object and referred to as if it were the object intended; often a lacuna is successfully used *procedurally* as the object intended; but it is not the *conceptual* object intended.

Although the presence of lacunas pertaining to distributivity of multiplication over addition may be apparent in the course of numerical and algebraic manipulations, we believe that sources of misunderstanding often run much deeper. That is to say, we anticipate and explore herein the possibility of lacunas within lacunas. Thus, we hope to unravel the 'harmful tangle' of meaninglessness with a more structured analysis of lacunas in order to determine and address difficulties that undermine and compromise the understanding of concepts such as distributivity.

Method

Participating in this research were preservice elementary school teachers enrolled in a course "Foundations of Mathematics for teachers" which is a prerequisite for teachers certification. In the midterm of the semester, clinical interviews were conducted with a subset of 18 volunteers from the class. Additional data have been drawn from the answers obtained from all 67 students of the class to selected problems on their final exam. The interviews were recorded and transcribed. During the interviews, each of which lasted for about half an hour, the participants were prompted, when appropriate, for understanding that may not have been apparent from their initial response. In circumstances where participants experienced difficulties with a particular question they were encouraged to reflect upon and articulate, or otherwise describe, the nature of those difficulties. In cases where such activity proved inadequate in leading the participant to a realization of a solution or, alternatively, a resolution of their difficulties, the interviewer would progressively allude to, or provide, additional information. This method proved conducive to uncovering and identifying the possible source and depth of conceptual and/or procedural difficulties. Finally, the data from the final exam were classified, compared and calibrated with the data obtained from the interviews wherever possible.

Synopsis of Results

Although the interviews were held in mid-semester, prior to the class' final exam, we will begin with the responses obtained from the latter as they provide a quantitative context with which to situate the more detailed qualitative data from the former.

Analyses of the responses (n=67) of two exam questions (can you represent A=15x5623+60 and $B=x^2y^5z+x^3$ as multiples of 15 and x^2 respectively) revealed that distributivity was explicitly used by 16 of the students for A and by 19 students for B. The 15 of the 16 students that explicitly used distributivity for A did so for B as well (Note: some upward bias is possible in these results as 7 of these 15 students explicitly using distributivity here were amongst those who previously participated in the interviews). The additional 3 students that used distributivity for B preferred to calculate their answers for A.

With respect to A, 23 out of the 67 students *calculated* (21 correctly, 2 incorrectly) to represent A as a multiple of 15 (that is, evaluated A and divided by 15 to obtain the multiplicative factor) while 14 students *inferred* that A was a multiple of 15 by observing that 15 divides both summands (however, distributivity was not used to represent A as a multiple of 15 in these cases). Two students inferred A was a multiple of 15 "because 15 is in the equation". One student simply asserted that 15 divided A, while another represented A-60 as 15x5623. Notably, 8 of the students either explicitly or implicitly assumed A=15x(5623+60). The remaining 2 of the students responded in a manner that offered no apparent meaning.

As calculation was not possible, the balance of the results with respect to B were more diverse. Nevertheless, some comparisons were possible: 16 of the 67 students inferred that either B was or was not a multiple of x^2 depending upon whether x^2 divided both summands (however, as with A, distributivity was not used to represent B as a multiple of x^2

in any of these cases). Indeed, 12 of these 16 incorrectly inferred the negative, denying that x^2 divided one or both summands. Similarly to the above, 6 students, (3 of which did likewise for A), assumed $B=x^2(y^5z+x^3)$. Responses unique to B included: 4 that instantiated constants for the variables, calculated and asserted the general case; 3 made explicit errors with exponents unrelated to distributivity; 2 asserted x^2 to be a multiple of B with no further justification; one unsuccessful attempt was made to divide B by x^2; one individual calculated the derivative of B with respect to x; and finally of the remaining students, 5 provided no response at all while 10 responses, such as "$x^2(1)+y^5z+x^3$" seemed too tenuous to categorize.

We used two sets of interview questions to pursue the understanding of relationships between addition and multiplication of those participating in this study. The question sets were designed with the assumption that a major source of conceptual difficulties with distributivity were most likely to be found in terms of multiplication (as repeated addition) and/or factorization (as forming a product from a sum).

The tasks of the first question set were based upon an arithmetic sequence, from which the first five members were provided: 3, 10, 17, 24, 31. We began the question set with a request for the 6th element of this sequence, subsequently requesting progressively larger elements of the sequence: e.g., the 9th, 20th, and 100th. At some point, in most cases, a formula for the nth element was requested. The purpose here was to determine the extent to which our participants had appropriately encapsulated multiplication as repeated addition. Half the group (n=18) responded in terms of sequential additions of 7, although most of these felt there was 'an easier way to do it'. In response to prompts to formulate this type of thinking we obtained: n=n+7 and #20=#19+7. In the latter case this participant noted "... but that doesn't help if you don't know 19 yet". The other half of our participants at some point illustrated an ability to multiply the difference and add on. The most typical strategy here was of the nature: f(20)=(20-9)7+f(9). One individual, John, did obtain the formula (k-1)7+3=n but was unable to apply distributivity when asked to simplify the expression, apparently because (without applying commutativity to the product) the symbols were not in a recognizable pattern. Most significantly, however, these interview data indicate that for half of our participants, the conceptual object of multiplication contained insufficient meaningful content to qualify as an adequate encapsulation. Andrea, apparently speaking on behalf of her classmates in elementary school, indicates that with respect to multiplication and the emphasis placed upon memorization and calculation "...we didn't really understand, I don't think we really understood why it worked". She reports that "... in my mind (thinking of multiplication as repeated addition) was pretty separate".

This may in part explain the difficulties our participants had with the second set of interview questions which approached distributivity in a more direct fashion. Here we provided the number A=3(9+4) and requested that A be expressed as the sum of two numbers. Only half of this group used distributivity to obtain A=27+12 while the rest calculated A=39 prior to providing the requested sum. Of the latter we asked if A could be readily expressed as the sum of two products from its original form: most were then able to recall the 'rule' and apply it. Our participants experienced much greater difficulty when we then provided the number B=12(147)+36 and requested they express B as a product of two whole numbers. Remarkably, 16 out of 18 calculated B=1800 prior to answering the question. When asked if they could express B as a multiple of 12 from its original form, at least 9 had to be prompted to consider 36 as 12(3). Of these, 3 subsequently attempted to apply the FOIL rule for the product of two binomials. Most others were then able to recognize that 12 could be factored out. Many of the participants had difficulty identifying similarities between the rule they had used with A and the one used with B. As one of our participants, Jill, expressed: "These (rules) are totally different, cuz this one you are factoring out the number and this one you are just multiplying them together".

Discussion of Results

The overall result of the responses to the final exam questions indicate to us that the level of performance and understanding regarding the use of distributivity may not have substantially improved for many students since they were first exposed to it in the middle grades. Assuming that 60 was readily recognizable as a multiple of 15, distributivity yields A=15(5623+4) in an effortless manner compared to the numerical calculation favoured by 34% of the class. Bottom line: 75% of this class did not explicitly recognize the appropriateness of factoring, let alone distributivity, in this situation. In the question involving variables, where numerical calculation was no longer an option, the overall performance of the class improved marginally by about 3%.

From the interviews, we have evidence that lacunas of the conceptually prior concepts of both multiplication and factorization are likely to play a major role in lacunas of distributivity. From both the exam questions and the interviews there is evidence of other lacunas as well. In particular, an significant number of individuals were either not aware of, or unconstrained by, the computational priority of multiplication over addition. That is to say: properly interpreting AxB+C as (AxB)+C, not Ax(B+C). Another set of difficulties that came to light in the interviews, such as those encountered by John with seeing (n-1)7 as 7(n-1) and others using the FOIL procedure for expressions like 12x147+12x36, indicates a reliance upon pattern recognition to provide queues for invoking a particular procedure. Such queues seem to be part of the procedures themselves. One of our participants put it this way: "...when I was learning this type of thing (distributivity) in school ... you knew you had to use that rule, so that my problem now, it's just, just recognizing it".

Our attempt to understand Jill's remark raised to our attention another major lacuna of distributivity, apparently sourced within the concept of equality: Given the procedural emphasis of calculation, it is quite natural to interpret '=' as denoting the causal outcome of a process as opposed a conceptual relation of equivalence. Indeed, the concept of equality is likely to remain a lacuna within arithmetical understanding without a realization that both sides of a numerical equation form different aspects of the same number. It is unlikely, however, that any lacuna is an island unto itself. As noted, 9 of those interviewed were unsuccessful in substituting multiplication for repeated addition in the problem situation presented in question set 1. However, 3 of these cases reveal some interaction with the lacuna of equality as an outcome of a process (which we will denote as '=>' in place of '=') in that when pressed to formulate a solution to the problem responded in terms of 'take the one before and add 7'. That is essentially to say f(n-1)+7=>f(n). Given that distributivity is usually presented as a(b+c)=>ab+ac, it is not surprising, assuming this lacuna of equality, that the process of factoring, i.e., ab+ac=>a(b+c) would be seen as 'completely different'. This lacuna may explain why we have so regularly observed a peculiar awkwardness with many calculations in our interviews; e.g., evaluating an expression such as 12(147)+36 as 12x147=>1764+36=>1800 ('bringing in' the 36) rather than (appreciating the redundancy of) 12x147+36=1764+36=1800. Indeed, because equality is so central to mathematical understanding, we believe that often, this lacuna remains deeply ingrained and forms a fundamental flaw in encapsulations of subsequent mathematical concepts such as Distributivity.

Conclusion

In this study we have found it useful to posit a theoretical construct that we have referred to as a lacuna. Our motivation for this has been to find a way to talk about, investigate and identify latent gaps in preservice teachers' understanding of mathematical concepts, using the elementary concept of distributivity as a case in point. This concept appears to be encapsulated as lacunas in various ways for many of the participants in this study and thus do not properly express the object intended. Apparently many significant lower order lacunas disguised as the proper objects of multiplication, factorization, equality, etc. are embedded within these lacunas of distributivity. That university students, after years of schooling in mathematics, should illustrate inadequate understandings of distributivity and its importance in relating the fundamental operations of addition and multiplication is disconcerting to say the least. Given that two thirds to three quarters of a university mathematics class were unable to recognize problem situations, readily solvable as an application of the distributive law, on a final exam, should be cause for concern.

Identifying a problem, however, must at some point give way to the problem of understanding why there is a problem to begin with. The major factor we see emerging here is the following: Emphases in early mathematics education are most naturally focused upon actions and processes associated with elementary concepts of arithmetic. At some point in a student's schooling, for whatever reasons, these elementary actions and processes are *assumed* to be properly reified or encapsulated as objects of conceptual understanding when in fact they are not. As the demands of mathematical learning comes to involve more complex and sophisticated concepts and procedures that presuppose understandings of the elementary concepts, they in turn are compromised. The result is a predictably fragile and riddled structure of understanding that may readily collapse like a house of cards, even when only presented with elementary problems in novel contexts. If this sketch of the situation is even close to the mark, the implication is that much stronger emphasis should be placed upon 'revisiting' the elementary actions and procedures of arithmetic in order to develop a proper conceptual understanding of them in the middle grades.

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ELEMENTARY TEACHERS' UNDERSTANDING AND IMPLEMENTATION OF UNITIZING OPERATIONS

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This report discusses one of the findings of a three week teaching experiment involving five prospective elementary teachers. The purpose of the study was to follow and describe their cognitive processes as they engaged in the formation of units. An attempt was made to gain an understanding of how they constructed units and whether or not their attention to and understanding of the unit concept would increase their understanding of rational number concepts and operations. Analysis of the data suggests that while unitizing occurred naturally in hands-on activities, there are certain cognitive obstacles that block the initial reaction of unitizing when presented with a worded problem. This report addresses those obstacles.

The difficulties with rational numbers encountered by children, as well as their teachers, has been well documented (Behr, Lesh, Post, & Silver, 1983; Hart, 1988; Kieren, 1976; Lamon, 1992; Mack 1990). Work by Behr et al. (1983) attribute these difficulties to a curriculum which has emphasized procedures rather than understanding.

In an effort to improve conceptual understanding of rational numbers, research has focused on informal knowledge and the unit concept. Informal knowledge: is acquired outside of, or in spite of school instruction (Lave, 1988; Mack, 1990); is related to real-life situations, constructed by students and brought to school instruction (Mack, 1990); and can provide a base for developing conceptual understanding (Mack, 1990; Pothier & Sawada, 1983). Von Glasersfeld (1981) suggests that knowledge of unit formation is intuitive or informal and Lamon (1992) contends that this intuitive knowledge of the unit concept can be used as a foundation for building rational number understanding.

Whereas the original study investigated the power of the unit concept in promoting rational number understanding, this report focuses specifically on the cognitive obstacles that appeared to inhibit the implementation of unitizing operations.

While there is no universal definition of unit established in the literature, throughout this study the word unit was used to describe a conceptual grouping of a collection into a whole; a cognitive entity. The unit concept encompasses the notions of *unitizing* (unit formation), *reunitizing* (unit decomposition), and *norming* (unit recomposition). Unitizing refers to a conceptual process of splitting or uniting one or more groups in order to form one or more units. Once units are formed, reunitizing occurs when the original units are restructured to form new units or composite units. Norming refers to the process of imposing a common unit structure on one or more units.

Methodology

The cognizing of units was observed in a three-week (six-session) teaching experiment designed to provide situations for exploration of the concept of unit. The experiment consisted of five group lessons and one individual interview. The lessons were designed to illustrate the three aspects of the unit concept. These lessons, as well as the individual interviews, were videorecorded and transcribed. Data sources included: video records of the six sessions, a journal consisting of reflections, spontaneous observations, questions, ideas, etc., and students' task sheets, homework, and essays.

Results

Analysis of the data revealed much about the conceptualization of the unit. The students' performances on the individual tasks clearly showed the formation of various types of units. That unit formation is intuitive (von Glasersfeld, 1981) or informal is supported by their individual comments. About the various units they were forming: "It just came naturally" (Mary Gail), "Naturally" (Renee), "No, I didn't mentally think about it" (Mary Gail).

While the informal nature of unit formation became evident, two obstacles to the understanding and implementation of the unit concept also became apparent: algorithm dominance and mathematical perception.

Algorithm Dominance

One of the biggest obstacles faced by the students while trying to conceptualize and extend the unit concept can be described as algorithm dominance. Through previous years of mathematics they had been exposed to and memorized numerous rules and algorithms. Their first reaction to a new situation was to recall previously memorized processes. When these processes could not be immediately recalled, they were lost because they had no conceptual system for determining an answer.

This algorithm dominance surfaced in lesson 2. The students were given worded problems that could be solved by unitizing to composite units or by using the more traditional approach which involved conversion to units of one through multiplication. For example, in one of the problems the students were asked to determine the number of snack-packs, which contained three donuts, that could be made from four dozen donuts. The units-of-one solution, which is traditionally the method of choice, involves converting the four dozen-donut units to 48 single-donut units and then dividing by 3 to determine the number of snack packs. The unit emphasized in this method is a single donut. Whereas understanding aspects of the unit concept might generate an alternate solution of determining the number of snack-packs per dozen and then repeating this four times. In this alternate approach, the unit of focus is a 12-donut unit. After all students used the traditional approach the following dialogue occurred:

Tena: OK,... like yesterday when everyone counted, everybody grouped. But when I gave you problems like this, almost everyone of you switched back to units of one...

Lisa: Easier. When you're counting hands-on there is more chance of a mistake... so if you group everything it's easier.... In a math problem, when you're doing it on paper, I think it's easier to figure it out in ones because that's the way we've always done it. I mean that's the way I learned to do it.

The above discussion suggests that the choice of the traditional units-of-one method for completing the tasks of lesson 2 was based on habits developed through prior instruction. This is further supported by the fact that all students were able to provide a correct solution using units other than one when asked for an alternate approach. So while the unit concept was conceptualized well enough at this point to provide a correct solution to these tasks, the more traditional approach was chosen because "that's the way we've always done it" (Lisa).

This pattern of algorithm dominance continued throughout the teaching experiment but was especially apparent in the teaching interviews. The following comments are taken out of context but reveal the algorithm dominance that surfaced when students were asked about rational numbers:

Renee: I think you are programmed. You always know that you have to convert whenever you don't have the same thing [refers to unlike denominators]. All those rules just kind of stick out.

Lisa: All I remember the teacher saying is that you could not say flip. You didn't flip the numbers, you had to take the reciprocal of the numbers. I never really knew what the difference was between the reciprocal and flipping.

Carolyn: I didn't know how or why you had to do it, I just did it...That's my basic idea of math.

Mathematical Perception

The other main obstacle, which might be considered an activator for algorithm dominance, was mathematical perception --how mathematical (as compared to real life) the student perceived the problem to be. If the problem was perceived as a math problem the unit structure was usually limited to units of one and algorithm dominance was present. The more realistic the problem was perceived to be, the more flexibilitydemonstrated in the unit structure. The obstacle of mathematical perception can be seen in the prom problem of lesson 4 in which the students were to plan the number and size of the tables needed to accommodate the various groups attending the prom -- 56 couples, 15 groups of four, eight groups of three, and six people coming alone. The students were reminded that those who came as a group or couple would want to sit together. The first reaction by the students was consistent with earlier reactions to problems perceived as math problems, they calculated the total number of people so they could divide (i.e., the units-of-one approach). Through the discussion the problem became more realistic to the students and this original approach was abandoned. Even upon finding a workable solution to this problem, the students made additional changes based on social aspects. The following dialogue occurred after the students realized that their current solution would leave one couple out of the couples only section:

Renee: Now, you may have a fight with these two people that are left over unless you just threw them in.

Mary Gail: Well, let's say everybody understands...

Renee: But you are saying that this couple can't sit in the couples only...

Mary Gail: If we go back to that, not everyone is going to sit at the same time, so do you really need that many tables?

Renee: But every girl needs a place to hang her purse. She doesn't want someone else sitting there if she goes to dance.

The above discussion provides evidence that the prom problem became more than just another routine math problem. The more realistic the problem became the more factors they needed to consider and the more flexible the unit became. It also illustrates a willingness to adjust the unit based on a social problem rather than a numerical one.

Discussion

Results of this study supported the contention that unit formation is informal knowledge but revealed certain cognitive obstacles in applying unitizing operations. In the hands-on activities of the first lesson, the notion of units other than one occurred spontaneously, but this notion was abandoned and replaced by the traditional emphasis on units of one when they were presented with a worded problem. Whether or not this behavior would occur with symbolic problems is suspected but not supported by this study. This abandonment of the initial instinct to unitize was explained by algorithm dominance and mathematical perception.

Further study is needed if we are to overcome the cognitive obstacles that block the use of the unit concept. The obstacle of mathematical perception indicates that the artificial nature of mathematics problems sometimes hinders the implementation of the unit concept. However, the data revealed that the students' discussions of the problems generated a more realistic interpretation, thereby providing a more intuitively grounded approach to the problems. This would suggest that one attempt to make current textbook problems more realistic might simply entail a classroom environment in which groups of students could openly discuss and interpret the situations presented in the problems. Regarding the dependence on previously memorized algorithms, algorithm dominance, it would be interesting to observe what, if any, obstacles would surface if the unit concept was emphasized before the algorithms. This was not possible in the present study since the students involved were at the college level. A study involving young children who are unobstructed by previously memorized processes might be beneficial in determining the power of unitizing operations.

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ELEMENTARY SCHOOL TEACHERS' PERCEPTIONS OF ALGEBRA: THE ROLE OF MODELLING AND TECHNOLOGY

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Throughout a 15 week algebra course, six teachers of grades K- 8 were asked to describe their views of algebra. The course was a study of functions derived from modelling of real-world situations where relationships were investigated through multiple representations in a technology rich environment. Teacher reflections were monitored through weekly journal questions, exam questions and student interviews. Case studies of six teachers investigate changes in perception of the objects used to describe and analyze relationships, and the role of modelling and technology in explorations. Within individuals, changes occur across time but the rate and path of these changes vary across teachers.

Introduction

This paper examines the perceptions of elementary and middle school teachers as they study a semester of algebra. Results are derived from their reflections as learners of algebra and their thoughts as teacher researchers when they interview their own students. The paper investigates teachers' views about the role of modelling and technology, and changes in the objects used for algebraic investigations.

Background Descriptions

The algebra course was a 15 week, 45 hour course developed through a study of functions, families of functions and relationship between functions. Families of functions were studied as needed in the development of models for applied situations and functions were used as objects for exploring and abstracting relationships both within and across models. Through out the course which used Computer-Intensive Algebra as a text, technology was used as an exploratory tool. The format of the class also deviated from that of a traditional mathematics course. Teachers worked in groups, presented the results of their explorations in written papers and oral reports, and even worked on exams together. Thus both the content and the classroom environment violated each of these experienced teachers' perception of what is and should be "Algebra" or even what should be a "Mathematics Class".

Before discussing the data, it is important to say a bit about these teachers. Each of the teachers was thoughtful, articulate, and dedicated. All of them had college degrees and course work beyond their college degrees which means that they had taken 3 to 4 years of high school mathematics, completed at least two college mathematics courses and taken at least one additional mathematics course as a graduate student in education.

Data Collection

In order to maintain an ongoing log of the reactions for each of the teachers in the class, a weekly journal was required. Specific questions were posed each week, a total of 30 questions across the course, which elicited responses about content, feelings towards algebra, and perceptions of their students' reactions to questions. Teachers were invited to add to or delete any questions which they were not ready to respond to. Although the instructor collected, reacted to, and returned the journals following each unit, the teachers kept their own journals and were able to review previous responses at any time. Teacher perceptions were also elicited through in-class discussion questions and exam question. For a final course project, each teacher was asked to interview at least one of their own students or a student from their school. Case study profiles for six teachers is built upon responses collected at the beginning of the course, following the midterm exam and the end of the course. Sample quotes are selected to either describe the thinking of the group or to trace the changes in an individual's perceptions.

Results

Usiskin's model of conceptions of school algebra and uses of variable as described in Usiskin (1988) provides a framework for analyzing teachers' perceptions of algebra. A summary of his four conceptions of algebra and corresponding uses of variable are included in the table below. Discussions of the meaning of variable draw as well on

Kuchemann's hierarchy (1981) in which he describes the six levels of understanding of variable as letter evaluated, letter not used, letter used as an object, letter used as a specific unknown, letter used as a generalized number, and letter used as a variable for a range of values.

The six case studies provide interesting contrasts. From the initial concept maps, through responses to weekly questions, conceptions of school algebra are traced in Pence (1994). A summary of the number of teachers whose primary perceptions of algebra place them in each category both at the beginning and end of the class is given below:

Conceptions of Algebra (Use of Variable)	Beginning of Class	End of Class
Generalized Arithmetic (Pattern generalizers)	2	1
Means to solve problems (Unknowns, constraints)	4	2
Study of Relationships (Arguments, parameters)	0	2
Structure (Arbitrary marks on paper)	0	1

Although classifications are difficult and open to debate, the table helps to illustrate that, across the semester, even though these teachers had taken four years of secondary school math, several college math courses and taught algebra themselves, the beliefs about algebra changed. These changes included views of algebra and uses of variables. Closely related to views of algebra and variables are the perceptions of the objects needed to analyze relationships and the perceived roles of modelling and technology.

What is the role of Modelling?

All of the teachers agreed that modelling helped improve their understanding. On the first day, the teachers were asked to discuss a major obstacle in their learning of algebra. Elisha's following comment is representative: *The biggest problem was that it meant nothing to me. The problems were not based on anything so I felt I was just suppose to memorize everything in order to get the right answer to a useless problem.* The end of the semester discussions of the role of modelling were also consistent with this statement by Judith: *Real situations are what made the course "come alive" for me I have* *always been the math student who muttered under my breath, "who cares?" I couldn't apply myself because I was frankly disinterested in a + b = c or b>a>0. When I had to factor equations I would turn off to algebra before I'd get to the "fun" part of creating table and graphs that displayed the relationships.* Thus, modelling gives a reason for developing algebraic understanding.

Modelling, expressing in mathematical form the relationships between quantities in the real world, was a central theme for the course. Ideas were introduced through modelling and individual understandings which resulted from the situation were discussed and used as the spring board for more in-depth mathematical explorations. Assigning a motivational role to modelling does not address what mathematical understandings were facilitated through modelling. To look at the role of modelling in the development of algebraic thinking, changes in mathematical objects used for analysis of the relationships needs to be examined.

What Mathematical Objects are used to analyze relationships?

Although there was general agreement that modelling made the study of algebra come alive, there were differences in what was perceived as the mathematical objects used to analyze and describe relationships. For three of the teachers, the objects of modelling were letters, numbers and arithmetic operations. Expressions were never viewed as an object. Joan's final description of algebra was the study of *numbers and letters in different operations to solve equation, word problems, polynomials and fractions*. Joan, however, struggled to resolve a conflict between the points of a graph and the graph as an object Following the final she reported that *I tried to think of a reason for us to move the curves around and couldn't.*

A fourth teacher struggled to change her perception of a variable and also change the object she used for analyzing graphs. Elisha was actually able to move from the letter used as an object, (she would only allow x as an unknown) to the letter used as a variable for a range of values. Graphically, she moved from points as objects to functions as objects. She was able to compare graphs but only began to explore operating on the functions. Following the final exam she wrote, *I liked working with tables and graphs* *before using symbols. I have always found tables helpful, and all of the graphing finally made algebra make sense to me. I especially liked seeing how different functions changed the graphs (linear, quadratic, rational, exponential).*

The remaining two teachers, Judith and Tom, by the midterm, perceived functions as objects of analysis. For example, Judith reported after the midterm that, *I was happiest when I had a function to use.* Tom in reflecting about the midterm stated that, *I guess the easiest problems were where the data was provided through graphs which seem easiest for me to understand and interpret and explain.* Although both Judith and Tom were investigating situations through the use of functions, their focus for abstraction differed. For Tom, modelling involved finding patterns from the graphic representation. Again following the final he stated: *I think the role of functions is to help us find patterns that may not be obvious.* For Judith, the goal of modelling was to *find the structure of a system and thus the control of the system..*

What is the role of Technology?

The role of technology is related to beliefs about the nature of the objects used in modelling. For this class, we used spread sheets, the TI 81 calculator and the IBM Tool Kit. To describe the role to technology, many factors need to be considered with the first component being the willingness to use technology. If teachers are willing to use technology as a tool, the question of the role of the technology then becomes linked to the teacher's view of algebra, modelling and the objects they use for analyzing relations. Of the six teachers, two refused to use any of the technology. This means that all tables and graphs were calculated by hand or by using a simple four function calculator.

For the remaining four teachers, the role of technology is varied and these variations are closely linked to the objects used for modelling. Each teacher was willing to use technology and in fact did use it as a tool for analysis on the final. Elisha initially found technology frustrating. During the seventh week of the class she stated that, *I don't like using technology because it takes me at least four times as long to do the problem than if I used paper. I find that I have to work the problem completely on paper before I can figure out what I'm suppose to be doing with the formulas on the computer or the TI 81.*

Elisha was struggling with a changing view of algebra. Although she saw technology as a complicating factor in the struggle, her search to explore relationships between functions forced her to persevere. On her final questionnaire, she described technology as very helpful and said, *It was really difficult at first . . . This was frustrating because it took so much time. I thought the TI 81 was amazing once I figured out how to use it.* On the final. Elisha seemed to use a calculator to develop all tables and graphs. She was working with the functions as objects and the technology was used as a tool for comparing objects.

Unfortunately, although Joan found technology easier to use at first, she used it selectively. For example on her final she would either use it to create tables or graphs but never both tables and graphs for the same problem. Thus if she used the technology to create a table, she would develop the graph by hand, plotting individual points. She described this sense that using technology was maybe something supplementary to the real math in the statement: *I liked using the calculator and computer playing around with them to look at different ways I could do a problem. Unfortunately, I didn't have time to explore.* Did she believe it was important to have time to explore using technology?

From the beginning, the remaining two teachers liked to explore using technology. They built their skills quickly and integrated technology as a tool in their investigation early. But, their reasons for using technology were quite different. Judith describes the role of technology as: *The power to jump over the "boring" chores of factoring and table-making and get to what interested me the most - seeing how thing fit together.* She wanted to think about connections, abstractions, and structure. Tom, on the other hand, who was interested in visual relationships, described his perception of the role of technology as follows: Technology *helps us visualize for better understanding. It helps us with tedious tasks of computation leaving more time for reflection and analysis.*

Discussion

Although changes occur, does the language support communication of these changing visions? These case studies give an abbreviated view of differences in the way teachers think and talk about algebraic explorations, the same words may carry different meanings. This function based approach enriched perceptions of the objects and tools which can be used in algebraic explorations, but did it develop a common base for understanding and talking about algebra?

A key instructional approach was that of discussing investigations, conjectures, and reflections. These communications were often frustrating. The same words and symbols meant different things to each individual. In this final quote, Judith talks about this issue as she reflects on her collaboration with Joan on a project. Judith said *Often my partner had a very different way of looking at the problem. We seemed to use semantics quite differently. I always needed to reword what my partner said so that it made sense to me. My partner wasn't comfortable with my rewording.*

The issue of varying perspectives of algebra was identified in the lead paragraph of the summary of the conference discussion for the conference entitled Algebra for the Twenty-first Century (NCTM, 1993) where, in the summary of the conference discussions, it was stated that there were no clear directions nor common visions shared by the invited participants and *that perhaps the most revealing fact about the conference was what it did not accomplish.* (p.6). Diversity is productive if we can communicate these differences. Does the use of modelling in a technologically reach environment aid or confuse communication about understandings and beliefs? How can bridges be developed linking perceptions and the language of "algebra"?

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LEARNING MATHEMATICS WHILE TEACHING[1]

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Abstract: *This paper examines cases of elementary grade teachers learning mathematics in the context of their own teaching—as they explore mathematics content they are using with their students, as they consider student strategies and representations that are new to them, and as they try to understand how students are thinking about complex mathematical ideas. We consider what teachers must already understand in order to do this and discuss implications for teacher education.*

It is widely recognized that in order to teach mathematics for understanding, teachers, themselves, whose mathematical experiences have been limited to traditional instruction, must understand the content more deeply than most currently do (Ball, 1991; Cohen et al, 1990; Schifter, 1993). Although some programs have provided opportunities for teachers to explore significant mathematics content (Lappan & Even, 1989; Russell & Corwin, 1993; Schifter & Fosnot, 1993; Simon & Schifter, 1991), it is unclear just what teachers need to learn in order to be able to support their students' constructions of rich mathematical concepts. In fact, it appears that the new mathematical understandings teachers must develop and the teaching situations they must negotiate are too varied, complex, and context-dependent to be anticipated in one or even several courses. Thus, teachers must become learners in their own classrooms (Ball, in press; Heaton, in press; Featherstone et al, 1993).

Learning in the context of one's own teaching is not simply a remedial measure. Rather, it is a component of the pedagogy itself and already requires of teachers a fairly sophisticated understanding of the discipline. That means that teacher education programs must concentrate on helping teachers to develop the skills they will need in order to learn mathematics in their own classrooms.

In "Teaching to the Big Ideas," a joint project of EDC, TERC, and SummerMath for Teachers, we are exploring teachers' developing mathematical understandings and their effect on instruction (Schifter et al., in preparation). Data include classroom field notes, audio taped interviews, and papers and journals written by the teachers which include reflections on episodes in their classrooms. The project began in the summer of 1993. Drawing from our work in "Teaching to the Big Ideas" during the 1993-'94 school year, this paper examines cases of teachers learning mathematics in the context of their own teaching and considers what teachers must already understand in order to do this.

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What Do We Mean By Learning Mathematics?

When we first asked teachers to identify episodes in their classrooms during which they learned mathematics, they were stymied. To many of them, "to learn" seemed to mean either the acquisition of completely new knowledge about previously unfamiliar mathematics or, perhaps, an "aha!" experience in which an idea is apprehended for the first time. We propose to extend our definition of learning mathematics to include a more ongoing and gradual process in which understanding of familiar content is deepened as one makes new connections and distinctions. A new representation of mathematical relationships may illuminate an aspect that was previously invisible even though that relationship was already "known" or "understood." An unfamiliar problem or context may highlight a mathematical idea in a new way, making one's thinking more problematic and causing one to think more explicitly about what was implicitly known.

It is in the context of this view of learning mathematics—a gradual building and deepening—that we offer glimpses of teachers engaged in learning mathematics in the course of teaching. As is the case when we observe student learning, we see only a slice, and often do not know the whole story. However, we believe these episodes, taken together, provide evidence of teachers as learners of mathematics in their classrooms. This paper considers episodes that illustrate teachers engaged in learning mathematics in three contexts: 1) exploring a mathematical problem or question embedded in content they are teaching; 2) thinking through students' representations and strategies; and 3) looking underneath students' confusions or excitement to consider mathematical structures.

1. Exploring Mathematics Content

One of the ways we see teachers learning mathematics is by engaging directly in mathematical content they are teaching their students. Before they begin a unit, teachers might explore a mathematics topic by reviewing several resource books, solving some problems, or discussing the issues of content with colleagues. But sometimes it is only after plunging into work with students that teachers identify mathematical issues that they want to explore further for themselves.

Meg gave her second graders a word problem that she had not investigated ahead of time: what are some possible combinations of 12 marbles if each marble can be one of three colors? Once she observed her students working on it, she realized the problem was more complex than it had first seemed and decided she needed to understand it better. At home, she worked on the problem with her husband. They began with 1 marble to find the number of combinations possible with 3 colors (3 combinations: 1 blue; 1 green; 1 red) then moved on to consider 2 marbles (6 combinations: 2 blue; 2 green; 2 red; 1 blue and 1 green; 1 blue and 1 red;

1 green and 1 red), 3 marbles (10 combinations), and so forth up to 12 marbles. She noticed that the differences increased by 1 (with 2 marbles, there are 3 more combinations than with 1 marble; with 3 marbles, there are 4 more combinations than with 2 marbles; etc.), and wondered what would happen with 4 colors. Meg pursued this problem for 12 pages in her journal, and then commented,

> I have a *very* hazy, in-and-out picture of *why* I can make that number of combos with 4 colors of marbles and why it grows the way it does as you add a marble--because by adding 1, you're adding all these other possibilities. I could NEVER explain it at this point. AND, I *don't* get the relationship between what happens with 2 colors, 3 colors, 5 colors beyond that it grows REALLY FAST. . . . [emphasis as in original]

In doing this work, Meg explored how the number of combinations grew systematically and began to try to describe the patterns of growth. Now that she had a deeper understanding of the complexity of the problem, she turned her attention back to the thinking of her students and the issue of how to proceed in her classroom:

> What are the implications for asking 7 year olds to work on a problem with 91 answers? . . . The kids who approached it randomly thought about 12-ness, and adding numbers, and three parts--and some kids went further. But now I want to ask them something more manageable to see if some "finish" and know they're done.

She chose to pose the following problem to her class: "There are 23 cupcakes for our good-bye show. Some have white frosting. The rest have green frosting. How many of each color frosting might there be?" This problem presents only two, instead of three, color possibilities, making it more possible for students at this age level to work on generation, comparison, and organization of combinations and to decide whether or not they have found all possible combinations. As the year continued, Meg continued to think about combinations and to pose related problems to her students.

2. Thinking through Students' Representations and Strategies

Another way in which we observe teachers learning mathematics is as they engage in mathematical thought *with* their students. Learning takes place as teachers are confronted by student strategies or representations that are different from their own. In assessing the reasoning of students' responses, especially when they are unfamiliar and unexpected, teachers think through the mathematics again for themselves in new ways, seeing new aspects of familiar content, expanding their own understandings.

Dorothy, a second grade teacher, expanded her view of the process of subtraction, when she observed one of her students develop an algorithm that, she later wrote, "I had never thought of, or even imagined before." Dorothy was working with Michael and his partner after he had made the familiar subtract-the-smaller-from-the-larger error in the problem 52-28, getting an answer of 36. As Michael began to rethink his solution, Dorothy expected him to work out something about "borrowing" a ten from the 50 and adding it to the 2 to make 12, which would match her own representation of the problem. But, instead, Michael invented a different method, "You take the 20 away from the 50 and get 30. Then you take the 8 away from the 2 which is minus 6. Then you take the minus 6 away from 30 and you get 24." Dorothy reported that she had to ask Michael to repeat his solution several times before she understood his method. She commented in her journal, "I've since read Connie Kamii's book in which she describes several common methods 2nd graders use to subtract in this situation and [Michael's] method was one of them. It still feels new enough to me that I have to think it through each time. It is definitely a case of my learning some mathematics from my students."

As another example, consider Ellen, who had been working on multiplication and division with her third grade class. In her work with them, multiplication was represented as repeated groups, while division was modeled as dealing out. These were models of the operations with which Ellen herself was comfortable. During one class, Ellen asked Kevon to illustrate 7 x 3 on the board. Kevon began by drawing 7 circles. He then drew one mark inside each circle, paused, and drew a second mark in each circle. Ellen recognized these actions as what she and her students usually did for a division problem. She was about to have Kevon sit down to give someone else a chance, but she caught herself, hesitating as she studied the board. She asked the class, "What is Kevon doing? It looks like what we usually do for division but let's wait and see what he comes up with." Kevon then added a third line to each of his seven circles and wrote "21" to the right of the equals sign in his equation. Ellen asked Kevon how he came up with 21 and Kevon explained, "I put 1 of each, 2 of each, 3 of each." Ellen asked, "What does that mean?" Kevon responded, "I got 7 of them with 3 in each." Ellen thought about this for a long moment before answering. At this point in the interaction, she was challenged by Kevon's representation to expand her own model of multiplication. She said, almost to herself, "Seven groups of what? Seven groups of three. Should we consider what he did right?"

Once she and the children decided Kevon's representation fit the problem, Ellen asked the class if anyone had still other diagrams. The staff observer noted that Ellen's acceptance of alternative representations was unusual for her. In response, students offered a greater variety of solution methods than had previously been the case during the group's study of multiplication and division. We surmise that her recognition of Kevon's unfamiliar representation—beginning with 7 groups and then dealing out marks until each group had 3, rather than the iterating of threes she and her class had been using—acted as a catalyst to her invitation to students to show their own ways of thinking about multiplication.

3. Delving Underneath Students' Reasoning: Looking at Mathematical Structure

In the third kind of example, teachers' reflection on what is problematic in students' reasoning leads them to rethink their own understanding of mathematical structures. Whereas in the second kind of episode, teachers are assimilating a student's unfamiliar representation or strategy into their own view of the mathematics, in the third kind of example, teachers move beyond their students' thinking. In the examples given above in the second section, teachers are, in a sense, learning the *same* thing as their students are learning. But in situations in this category, teachers use student thinking as a catalyst. By probing underneath students' confusions about mathematical ideas, they confront new mathematics themselves as they ask: why would a student think that? what is right about the solution from the student's point of view? what is difficult to understand about the mathematics here and why might that be the case? Often this process involves making explicit and re-examining what they have implicitly known.

For example, students in Sylvia's second-grade classroom were adding 2-digit numbers, using base ten blocks to model "carrying." Most students easily undertook this task, following the conventions established in the class for using the blocks and recording their results. However, when some students were asked by a staff observer to determine without trading what quantity was represented by 4 tens sticks and 15 small cubes, they had difficulty counting this quantity. One student counted 10, 20, 30, 40 for the tens, then continued counting the ones by tens, and was perfectly satisfied with her result of 190. Sylvia was quite surprised when she observed students counting ones as tens. She had assumed from their competent use of the base ten blocks to solve addition problems that they understood how to decompose 2-digit numbers into tens and ones and how to recombine them. As she observed her students, she began to consider how the conception of a unit does not inhere in a particular physical representation (such as base ten blocks) but in a complex mental model. She pondered more deeply what is involved in constructing and simultaneously manipulating more than one unit (e.g., ones and tens).

This third kind of episode has been more difficult for us to capture. It consists of longer-term reflection, rather than learning that happens in a single episode. Throughout the year, Sylvia continued to try to define more clearly for herself the nature of a base ten system and to describe what "understanding" tens and ones requires. In May, she described how various students in her classroom were solving 2-digit addition problems, some breaking apart numbers into tens and ones, others counting on only by ones. She commented: "What goes on in someone's brain to make sense of tens and ones? Does a person need to construct a system of ones AND a system of tens that sort of 'fits' like an overlay on the system of ones? How does this happen?"

What Enables Teachers to Learn Mathematics While Teaching?

As we collect and analyze episodes such as those in this paper, we are beginning to identify some of the elements that appear to be necessary for teachers to learn mathematics while teaching. First of all, this kind of exploration of mathematics content requires teachers to see themselves as adult learners of mathematics and to see their own classrooms as contexts in which they learn. The teachers described here have participated in at least one year of experiences which emphasized adult mathematics learning, and they now share some assumptions: 1) that learning about mathematics occurs when you are immersed in problem solving; 2) that an important aspect of mathematical thinking is identifying, describing, and testing patterns and relationships; and 3) that understanding the mathematics better has implications for their pedagogical decisions. Rather than expecting to acquire discrete bits of information, these teachers assume that the way to learn mathematics is to do mathematics.

In the first category of learning mathematics while teaching, teachers explore the mathematics content in which they engage their students. In order to do so, they must be curious about mathematics; they must know that they can pose their own mathematical questions; and they must assume that they can pursue those questions themselves.

In the second group of episodes, teachers expand their understanding of mathematical ideas by paying attention to student strategies and representations. This requires teachers to develop a classroom culture in which multiple strategies and unexpected responses are the norm, where the point is not to find the solution prescribed by teacher or textbook, but to *reason cogently* about mathematical relationships. What the teachers expect for their students—that they develop and compare strategies—they also make use of themselves as a way to deepen their mathematical understanding. To do so, teachers must learn how to follow a mathematical argument and assess its validity.

In the third category of episodes, teachers delve beneath students' efforts at understanding to confront the underlying mathematical structures with which their students are grappling. Through carefully listening to and observing their students, they begin to identify and describe the complexity of elementary mathematics—what mathematical ideas are central to student understanding and why these are hard for students. When teachers reflect on student learning in this way, they are doing more than understanding student thinking better; they are actually recognizing and articulating significant mathematical ideas and developing a deeper understanding of these ideas for themselves.

Do teachers know that they are learning mathematics as they teach? In some of the episodes recounted here, teachers seem to be aware that they are engaged in mathematics learning; in others, it may be only the authors who view the event in this way. Often, we have seen that the simplicity of the formulations brought about in such new learning situations lead teachers to discount or disparage their new knowledge: "Oh, I was so stupid not to see that," or "Of course, I'm sure it was obvious to everyone else, but . . . ." On the other side of learning, the simple and elegant might appear trivial—not worth mentioning. Beginning to notice and appreciate the profundity of our own insights, as adults, is connected to appreciating the profundity of children's insights—insights which might also seem to an adult trivial and obvious.

The dispositions, understandings, and skills that underlie the kind of learning described in this paper are not cultivated in a traditional mathematics curriculum and are not supported in conventional school cultures. One of the objectives of teacher education programs should be to prepare teachers for the ways in which they can take advantage of their teaching as a site for their own ongoing learning of mathematics. Making explicit and validating new learning of this kind may be a critical function of teacher education.

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TEACHERS' CHANGING CONCEPTIONS OF THE NATURE OF MATHEMATICS: A DEVELOPMENTAL MODEL

by Deborah Schifter, EDC

This paper distinguishes four conceptions of mathematics as enacted in the classrooms of teachers working to transform their instruction along lines urged by the reform movement. Drawing on teachers' accounts of their own teaching, it proposes these enacted conceptions as four stages of a typical developmental trajectory. Questions concerning the implications of this model for both a theory and practice of mathematics teacher development are raised.

Last year at PME-NA, Lynn Goldsmith and I (1993) outlined the components of a developmental model for teacher change in the context of the current mathematics reforms. We suggested that a model must address qualitative changes along three strands of practice: 1) activities, materials, and classroom structures, 2) epistemological beliefs as enacted in practice, and 3) mathematical understandings as enacted in practice. In devising and employing a way of thinking about typical trajectories of teacher change involving the first two strands, Martin Simon and I relied on the Levels of Use instrument developed by Hall et al. (1975) to analyze teachers' use of innovative strategies, and we designed a new measure to chart development of a constructivist epistemology as the basis of instructional decision making (Schifter & Simon, 1992). In this paper, I describe successive stages of a developmental trajectory against which the third strand may be assessed. (See Figure 1.)

I characterize conceptions of mathematics teachers enact in practice as: 1) an *ad hoc* accumulation of facts, definitions, and computational routines; 2) student-centered activity, but

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Strategies (Levels of Use) (Hall & Loucks, 1975)	Epistemology (ACMI) (Schifter & Simon, 1992)	Enacted Conceptions of Mathematics...
Does not know about or has rejected use of the strategy.	Does not hold a constructivist epistemology.	As an ad hoc accumulation of facts, definitions, and computational routines.
Uses the strategy; struggles with problems of classroom management with respect to the strategy.	Has a rudimentary understanding of constructivism, but has difficulty basing instruction in this understanding.	As student-centered activity, but with little or no systematic inquiry into issues of mathematical structure and validity.
Implements the strategy with ease.	Holds a constructivist epistemology and is comfortable with instruction, but focuses on teaching behaviors.	As student-centered activity directed toward systematic inquiry into issues of mathematical structure and validity.
Fine-tunes strategy to meet the specific needs of students.	Focuses on students' learning from a constructivist perspective.	As above, but systematic mathematical inquiry organized around investigation of the big ideas.

Figure 1. Three strands of the teacher development process.

with little or no systematic inquiry into issues of mathematical structure and validity; 3) student-centered activity directed toward systematic inquiry into issues of mathematical structure and validity; or 4) systematic mathematical inquiry organized around investigation of "big" mathematical ideas. Each conception or stage entails an understanding of what counts as "doing mathematics," of the extent to which mathematical results are interconnected, and where mathematical authority resides and how it is established.

These stages were derived from my observations in classrooms both of teacher-participants in projects of my own and of teachers in projects on which I consult, as well as from reports of classroom observations written by teacher educators who consulted on my projects (Schifter, submitted for publication). For this paper, I selected illustrations from teachers' own accounts of their teaching and mathematical thinking. They were produced for various teacher development projects in which I have worked since 1985.

The first of the four stages characterizes the mathematics of the typical contemporary U.S. classroom: facts and procedures, discrete and ungrounded, learned mechanically and mechanically applied. How authority over mathematical knowledge is bestowed is not an issue--truth is simply incarnate in textbook and instructor.

The introduction of innovative tools and strategies--use of manipulatives, group work, or electronic technology, for example--may be energetically promoted, being often mistaken for significant instructional change. But these are here used simply in support of traditional goals--as in the case of Ms. Murray, who describes how she uses manipulatives in her mathematics instruction:

> With division, first I do teach them a formula. . . . . And I do tell them that I want them first to look at the problem, ask themselves what is the smallest number the divisor can go into, put a check over that column. Because otherwise, they don't line up numbers correctly and don't even realize how many digits the answer is expected to have. But then, in using the manipulatives, I would say, well, sometimes if you start in a tens column for that first digit, that means that we're going to have to do some regrouping. So we do that with blocks. And then eventually we actually get into doing the division on paper without using manipulatives.

Ms. Murray's division unit is organized to teach procedures--with manipulatives or on paper, it makes no difference--which she demonstrates and hopes her students remember.

School mathematics at the second stage is characteristic of classrooms whose instructors are committed to encouraging their students' cognitive autonomy. Now, finding patterns, solving problems, making conjectures are all seen as central to the "doing of mathematics," and grounding these activities in familiar contexts--by using descriptions of real-world situations or introducing techniques of physical or diagrammatic representation--is understood to be an essential support for student initiative. Unfortunately, however, these instructional ambitions are compromised by a constricted view of the nature of the discipline and/or an inadequate grasp of disciplinary content. Mathematical actions and their results remain discrete when teachers are themselves either unable to recognize or frame deeper conceptual issues or don't see that this is where their teaching should lead.

For example, Mr. Smith describes how he used the Geometric Supposer with his high school geometry class. Students were asked to collect data and formulate conjectures about isosceles triangles. However, once their conjectures were listed on the board, he was unsure how to proceed. "Some of the conjectures were good, but others, I had to laugh. Like, all isosceles triangles are acute." When asked how he ended the lesson, he reports that he told the class which of their conjectures were correct, and which were incorrect.

Mr. Smith's colleague, Ms. Peterson, found herself in a similar situation when she experimented with the Supposer. Toward the end of a lesson, she, too, had a list of conjectures on the board--some of which were correct, others incorrect. However, since she did not want to discourage her students, she ended class without addressing the correctness of their conjectures and never referred to them again.

A dilemma facing teachers operating out of the disparity between their enlarged ambitions for their students and their own limited mathematical resources concerns how mathematical authority--validity--is to be established in their classrooms. The dilemma is this: if students are supposed to figure things out for themselves, what happens when they get it wrong? Rather than shift the locus of mathematical authority to the discursive process itself, teachers like Mr. Smith retain that authority. Other teachers, like Ms. Peterson, resolve the dilemma in an opposite, though equally mechanical way: if student autonomy is the goal, then all solutions are valid. Rejecting the tradition that accorded them cognitive omnipotence, but unable to envision for their classrooms a non-authoritarian alternative by which mathematical validity might be assessed, they are left presiding over a kind of mathematical anarchy.

In the third stage, the disparities of the second stage have been rectified--teachers now possess an understanding of the nature of mathematics commensurate with their aspiration toward a practice emphasizing student construction. Solving problems is no longer an end in itself, but a means toward the exploration of mathematical connection, of underlying mathematical order. When students announce the discovery of a pattern, this is followed up with questions about whether the pattern will always hold and how that can be shown or what information the pattern conveys. And the formulation of conjectures leads to the investigation of their validity. Ms. Jackson provides an example of a teacher who is learning to look beneath numerical patterns to explore the structure of the number system:

> My confidence in math has always been rather high. In high school and college, I was able to memorize formulas, manipulate and computate [sic] numbers; do what I saw as mathematics. With help from my students, I have learned that I was just looking at the tip of the iceberg. Mathematics is much deeper than this superficial view. There is so much more to explore below the surface.

To illustrate her point, Ms. Jackson describes how she distributed sets of base-5 blocks to her class and asked her students to create a number system using only the symbols A, B, C, D, and 0. Several students created the base-5 analog to the ancient Egyptian system: A = 1, B = 5, C = 25, D = 125, and 0 = zero (e.g., CCA = 25+25+1 = 51). She writes:

> One group noted that by just counting the number of symbols, you could determine whether a number was odd or even. For example, BA and CD are even because they have two symbols. DCBAA and CBB are odd numbers because they have an odd number of symbols.
>
> This puzzled me. Based on my old perception, odd and even were simple concepts to understand. Even numbers are numbers that can be divided exactly by 2--numbers ending in 0, 2, 4, 6, and 8. Odd numbers are not even and end in 1, 3, 5, 7, and 9. How could even and odd numbers be more than that? What else was there to know?

But if Ms. Jackson didn't yet know "what else there was to know," she was confident that if she listened to her students, she and they, together, could figure out just why counting the symbols in a numeral in their base-5 system was sufficient to establish whether that numeral represented an odd or even number. She began by asking her students to tell her what they knew about evens and odds. Some students offered definitions, others described the familiar pattern (even numbers end in 0, 2, 4 and so on), and still others talked about the results of operating with odd or even numbers.

> Clearly, my students saw even and odd numbers differently than me. Although some ideas were similar to mine, they had looked below [the tip of] the iceberg to see something more: relationships between odd and even numbers.
>
> We continued to explore these relationships, and I saw that in [their] numeral system, all the symbols have an odd value: 1, 5, 25, 125. When one combines two odds, you get an even number. When three odds are added together, you get an odd number. This is why it worked! It felt really powerful to be able to figure this out. My students and I had seen something new and wonderful. Odd and even numbers were more than simple patterns but concepts that impact number system structures and the relationships between number values.

At the fourth stage, teaching is organized to allow students to confront the "big ideas" of the mathematics curriculum. By "big ideas" I mean central organizing principles of mathematics with which students must wrestle as they confront the limitations of their existing conceptions. Through our work as mathematics teachers and teacher educators, my colleagues and I (Schifter, Russell, & Bastable, in press) have found that there are particular themes--themes that embody critical mathematical concepts--that arise time after time with different groups of learners when instruction is organized around and responsive to student thinking. It is by listening to students, identifying common areas of confusion or questions that intrigue, and then analyzing the underlying issues that the big ideas emerge.

For example, one frequently hears questions and comments like the following from students engaged in mathematical explorations: "I solved the problem by adding [on] and he solved it by subtracting. We got the same answer, but he did it wrong." "How can that piece of cake be 1/2 and 1/4 at the same time?" "We just said that this rectangle is bigger than the other one. Now you say it's smaller. That doesn't make any sense!"

When statements such as these are repeatedly heard, indicating persistent difficulties and confusions, points at which students need to step back and take time to ponder, they likely flag crucial conceptual-developmental issues. The big ideas which may be implicated in the quandaries of these students include: operations are systematically related and some problems might be solved using either addition or subtraction; the same quantity can be represented by different fractions, depending on their reference whole; an object or process may have attributes that change at different rates--the area of a rectangle, say, may grow while its perimeter shrinks.

Teaching to the big ideas would orient the study of the topics of the mathematics curriculum toward the major conceptual unities that run through school mathematics. Developing such a practice would mean learning to recognize the specific and pervasive ways in which attempts to construct instances of these unities go wrong. And rather than explaining what the big ideas are, it would mean giving one's students opportunities to confront the limitations of their extant conceptions and time to work through their confusion to construct new, more inclusive understandings.

For example, with each expansion of the domain of number--to include 0, or fractions and decimals, or negative or irrational numbers, and so on--the meanings of the basic operations must be rethought. So when Joanne Moynahan (in press) began a unit on fractions with her sixth graders, she knew they would need to reconsider generalizations they had made from their work with whole numbers. To start, she gave them time to work on the following problems using whatever strategies they chose:

1. The Davis family attended a picnic. Their family made up 1/3 of the 15 people at the picnic. How many Davises were at the picnic?
2. John ate 1/8 of the 16 hot dogs. How many hot dogs did John eat?
3. One-fourth of the hot dogs were served without relish. How many were served without relish?

After working in pairs for some time, the class came together to share their solutions. She describes what happened next:

> As we discussed each problem I recorded a shortened version on the dry-erase board. . . . At the end of sharing the board looked like this:
>
> 1/3 of 15 = 5
> 1/8 of 16 = 2
> 1/4 of 16 = 4

We didn't have much time left before the recess bell, but I thought I would . . . give them something to think about and posed the following question:

> Does anyone know what they were doing with these numbers? (Long pause.) What operation did you use? Did you add, subtract, multiply, or divide? (Another long pause.) What symbol could we put in here instead of "of"?

And so began a two-day conversation in which students pondered the actions they had taken to solve 1/3 of 15 = 5, analyzed different ways of representing those actions, and considered whether any of the symbols--"+," "-," "x," or "÷"--could meaningfully replace "of."

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Before turning to a series of concluding remarks--questions mainly--a note of caution needs inserting. Although I have drawn on teachers' accounts of their teaching in my characterization of four kinds of enacted conceptions of mathematics, one may not assign teachers to one or another of these categories on the basis of such slim evidence. Even teachers who tend to organize their instruction around the big ideas may choose, on a given day or in a given instance, to have their students practice computational routines.

At this stage in the development of an assessment model, many questions remain unanswered. For example, we find different degrees of affinity or independence among the various positions along the strands. Thus, Ms. Murray's description of her division unit embodies both the traditional view of the nature of mathematics--an *ad hoc* accumulation of facts and algorithms--and the equally conventional belief that learning is simply a matter of remembering. A rudimentary understanding of constructivism accompanies Mr. Smith's and Ms. Peterson's emphasis on student conjecture in their practice. And Ms. Moynahan's ability to listen to her students--the product of her maturing constructivist perspective--allows her to identify an opportunity to teach to a big idea. While Ms. Murray's case implies that a teacher can progress along the "strategic" strand independently of the other two, the couplings of epistemological with mathematical positions just listed suggest a more complex relationship between those two strands. Indeed, are these relationships so intimate as to call into question a model that distinguishes them?

While I have defined the mathematical strand in terms of the nature of the mathematics teachers enact in their classrooms, how is this related to their content knowledge? On the one hand, Mr. Smith's and Ms. Peterson's failure to thematize the issue of validity with their students may follow from limited content knowledge. And along with Ms. Moynahan's more sophisticated understanding of the nature and purpose of mathematical activity is a richer knowledge of the connections among concepts, representations, and real-world contexts and an appreciation of how mathematical truth is pursued and how validity is established. On the other hand, Ms. Jackson shows us how teachers can extend and deepen their grasp of disciplinary substance as they work with their students at levels three or four. The key here seems to be the degree of confidence they possess in their own powers to listen to and evaluate student reasoning (Russell, et al., 1994).

The questions continue: Does the scheme laid out in Figure 1 really map a logic of teacher development with respect to contemporary efforts in mathematics education reform? Does it allow us to identify "orderliness" in the process of teacher change? If so, what are the mechanisms by which teachers move from one position to the next? And how are teachers' individual motivations and dispositions involved? (Goldsmith & Schifter, 1993)

Finally, and the real point, is such a model useful to those of us working to reform mathematics education? That is, does it help us to define goals and to analyze our efforts with respect to those goals? Does it allow us to orient ourselves, as teachers, as curriculum developers, or as teacher educators, in order to determine next steps? Or does such a model tend to calcify our work--limit the richness of our analyses or confine the possibilities we might pursue?

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The Fundamental Theorem of Arithmetic: Used and Confused

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This study investigates preservice teachers' content knowledge in the domain of elementary number theory, with a focus on the prime decomposition of natural numbers. Participants' responses to the interview question and written assessment show that often the existence more than one prime decomposition is assumed. Drawing from the students' difficulties raises instructional implications to help students understand and apply the Fundamental Theorem of Arithmetic.

The Fundamental Theorem of Arithmetic claims that decomposition of a composite number into its prime factors exists and is unique except for the order in which the prime factors appear in the product. This theorem is a part of the core mathematics curriculum for preservice elementary teachers in many institutions, even though the formal mathematical proof of the theorem is usually not provided for this audience. On several occasions we've noticed that many preservice elementary school teachers are familiar with the theorem, may articulate and explain its meaning, but fail to apply it in various problem solving situations. The purpose of our study was to investigate this specific phenomenon in order to contribute to an improved pedagogical understanding of constructing knowledge of natural numbers and their multiplicative structure. Following Freudenthal (1983, pp. 112-3) we use the term 'multiplicative structure' to refer to "the whole of the relations (of all possible products) ... and all one can think about in this context". In particular, we will be thinking about products pertaining to the decomposition of natural numbers into their prime factors.

Background

This study is a part of the ongoing research on teachers' content knowledge in mathematics. One of our specific foci in this investigation is teachers' understanding of topics in elementary Number Theory. In previous research we have investigated topics of divisibility and factorization (Zazkis and Campbell, 1994). Extending our research to topics involving the Fundamental Theorem of Arithmetic is not an isolated goal in and of itself, but rather a natural progression of our investigations in this area.

Conceptual Framework

Several overlapping ideas about the development of mathematical knowledge and its nature form the conceptual framework for this study. A repeating theme in the literature is that of the formation of cognitive structures (Piaget, 1972) or conceptual entities (Greeno, 1983). More recently such conceptual entities have been addressed in terms of mathematical or mental *objects* and similar, though not identical, ideas regarding object construction have been proposed by various authors. (Artigue, 1992; Duady, 1985; Dubinsky 1991; Sfard, 1991). A common theme highlights the importance of object construction, referred to as *encapsulation* (Dubinsky) or *reification* (Sfard), in mathematical understanding.

Of these theories we have adopted Dubinsky's action-process-object-schema framework (Dubinsky, 1991; Dubinsky et al., in press) with the following qualifications and considerations in mind: (1) we have found it to be particularly conducive for preliminary analyses and interpretations of our data, (2) we wish to maintain lexigraphical and conceptual consistency with our previous studies on number theory related concepts and, (3) eventually, we hope to fully explore the strengths and limitations of this framework in the overall context of elementary number theory concepts, prime decomposition being but one amongst them.

Method

The participants in our research were preservice elementary school teachers enrolled in a course "Foundations of Mathematics for teachers" which served as a prerequisite course for teacher certification. The data acquired is based on written responses by 54 students on 3 assessment questions and subsequent clinical interviews with 21 volunteers from the same class. Both questionnaire and the interviews were conducted after the topics of elementary number theory were 'covered' in the course. This included topics such as prime and composite numbers, factor trees, prime decomposition and Fundamental Theorem of Arithmetic, divisibility and divisibility rules, greatest common divisor and lowest common multiple, among others. The assessment questions were administered on 3 different occasions within a 2 week period of time and the participants were not temporally constrained in addressing them. In these questions the students were invited to:

(1) consider the number $M = 3^3 \cdot 5^2 \cdot 7$ and decide whether it is divisible by each of the numbers 7, 5, 3, 2, 15, 11, 9 and 63.

(2) consider the number $K = 16199 = 97 \cdot 167$ (where 97 and 167 are given as prime numbers) and decide whether K may be divisible by 3, 5, 11, 13 and 17.

(3) look at a list of 15 numbers, including 8^2, 17^2, 17^3, 234^3, 234^6, $5^2 \cdot 17^2$, $5^3 \cdot 17^2$, $5^6 \cdot 7^2$, P^3 where P is prime, C^3 where C is composite, and decide which numbers on the list are, or could be, "perfect squares".

In the second question the participants were explicitly asked to try and answer the question without calculators or paper and pencil calculations. In the first and the third questions there were no specific instructions given regarding the strategy for solution.

The interviews, as part of a larger study, included a broad variety of questions related to topics in elementary number theory, with a focus on concepts like factors, multiplies and divisibility. The interviewer followed in depth on participants' responses and focused on justifications for their decisions. The interviews were transcribed and analyzed in terms of students' explicit or implicit use or apparent ignorance of the Fundamental Theorem of Arithmetic in their responses to the various questions, and any indications of their beliefs regarding the nature of prime decomposition were noted.

Results and Interpretations

Our participants indeed acquired knowledge of the Fundamental Theorem of Arithmetic: they could quote it when asked to and illustrate their knowledge by, for example, decomposing a "big" number like 360 via different routes and ending with the same (unordered) list of prime factors. On the other hand, the examples of successful application of the Fundamental Theorem of Arithmetic were limited.

On question (1) about one half (25 out of 54) students preferred to calculate M and then check for divisibility by either performing division or applying specific divisibility rules. A majority (29 out of 54) of the participants were able to accompany their positive answer regarding the divisibility of M by 7, 5, and 3 with a mathematical justification "since those are among the factors in prime decomposition". On the other hand, 16 out of these 29 were unable to apply similar reasoning regarding divisibility by 2 and 11, that is, to claim that those primes were not among the factors in prime decomposition of M. They regressed to calculation and only after carrying out division inferred that 2 and 11 were not among M's divisors. Participants' responses suggest that, for these 16 students a possibility of the existence of another prime decomposition was not over-ruled.

In question (2), 15 out of 54 students claimed, applying specific divisibility rules, that 16199 was not divisible by 5 or 3, but MAY be divisible by 11, 13 or 17. For many students the only way to find out (divisibility by 11, 13 or 17) was to perform division. Even though the number 16199 was presented to students in its prime decomposition (97 x 167), it seemed, again, that at least implicitly our participants believed that an alternative prime decomposition might be possible. To exemplify an explicit case, we quote Lesley's comment in this regard: "because a number is divisible by two primes does not mean that it is not divisible by other primes".

39 (out of 54) students claimed that 16199 was not divisible by any of the numbers on the list, that is, 3, 5, 11, 13, 17. The arguments to justify this claim varied. Only 13 students used the idea of uniqueness of prime decomposition, either explicitly, "the prime decomposition of 16, 199 is 167 x 97 and these are the only two prime divisors", or implicitly, "for any of the following primes: 3, 5, 11, 13, or 17 -- to be a factor of 16199, they would have to be part of its prime decomposition". On 3 worksheets explicit long division was carried out (calculators were not allowed on this question). Five students made the claim without providing any additional justification. Three students accompanied their correct decision with illogical arguments, such as: "these numbers will

not divide 16199, because 16199 is a product of two prime numbers, forming a prime number itself".

The remaining 15 participants (out of the 39 answering correctly) claimed that 16199 cannot be divisible by 3, 5, 11, 13, 17 because "167 and 97 are both primes and cannot be decomposed any more". That is, the focus of the claim was not on 3, 5, 11, 13, 17 being non-factors of 16199, but on 3, 5, 11, 13, 17 being non-factors of 167 and 97. The students making this claim are likely to have been thinking of prime decomposition as a factoring *process*, rather than the *object* of a number expressed as a product of primes.

On the third question it seemed necessary for about 1/3 of the students to calculate for example the value of 17^3 and then punch the 'square root' button on the calculator, in order to infer that 17^3 was not a square number. On the last two parts of question (3) – is p^3 a square number, when p is prime and is C^3 a square number, where C is composite – 49 out of 54 students responded exactly in the same way, making no discernible distinction between prime and composite numbers. Similarity in explanations demonstrate that students were treating C as one entity and not taking into account the prime decomposition of C. Inability to distinguish between the two cases by the overwhelming majority of students suggests that the students' construction of concepts of prime and composite numbers requires further refinement.

In our interviews, we found some explicit beliefs regarding the nature of prime decomposition. A belief that was evidenced in four interviews is that decomposition into prime factors means decomposition into small prime factors. This belief is demonstrated in the following excerpts with Lisa and Tanya:

Interviewer: Um, let's take the number 391. Would 391 be divisible by 23?
Lisa: . 23, (pause), I don't know. I don't think so.
Interviewer: Hmm, and why do you think not?
[..]
Lisa: 391, I think, is a prime.
Interviewer: And why do you think that 391 is a prime?
Lisa: Because I don't think it had been divided by 2 or 3, or 5, or 7 (laugh). . .
Interviewer: (laugh)
Lisa: Like I'm going through my mind with primes, and I know that it's not divisible by 2 because it's not an even number. I know it's not divisible by 3, because the sum isn't um divisible by 3. Four, same

reason for the 2, it's not even. Five, it has to be a 5 or a 0, um, 7, it's not divisible by 7.
Interviewer: And, what would be the next prime?
Lisa: 11.
Interviewer: You think it's divisible by 11?
Lisa: No.
Interviewer: And uh 23?
Lisa: No. I think it's a prime number.
Interviewer: Okay. So if you were to. . .
Lisa: I would have to figure it out the long way. . .

"The long way" for Lisa here means keep dividing by primes. It is interesting to note that she stopped at 11 to infer the primeness of 391, and didn't even try to divide by 23, in order to obtain a direct answer for the interview question. Tanya, as well as Lisa, conjectured that 391 was a prime number. To the interviewer's request for justification, she replied:

Tanya: I don't know. I guess, like I, um, like I was saying with, I know there's a way to do it, prime factorization, and I know that 23 is a prime number, but I guess, um, I was assuming, for some reason, that as long as 391 was not a prime number, it would have a factor smaller than 23, a prime factor
[...]
Tanya: I guess it's probably just more experience than anything, but it just seems to me that when you factor a number into its primes, I mean what you're doing is, you're trying to find the smallest, I mean numbers that can no longer be broken into anything smaller aside from 1 and itself, so that, I guess it's just the whole idea of factoring things down into their smallest parts. . .
Interviewer: Um hm.
Tanya: I guess gives me the idea that those parts are themselves going to be small.

It is interesting to note here, that Tanya's belief that prime decomposition is decomposition into "small" primes co-exists with her awareness of existence of very large primes.

Summary and Conclusion

Sfard pointed out the difficulty in reification in general and suggested that, at some particular point or level, it may be "out of reach" for some learners (Sfard, 1992). We have illustrated this difficulty in the case of prime decomposition of a composite number into its prime factors. While the *existence* of prime decomposition may be taken for granted, the *uniqueness* of prime decomposition appears to be counter-intuitive and often a possibility different prime decompositions is assumed. Since the proof for the Fundamental Theorem of Arithmetic is most likely to be omitted in basic mathematics courses for preservice elementary school teachers, we feel that some pedagogical alternative is needed as compensation. We believe that a proper understanding of the concept of prime decomposition is central for the understanding of the structure of whole numbers and that specific instructional techniques are required to help students revisit and reconsider their intuitive beliefs. Exploring the dimensions of these intuitive beliefs with activities based upon assessment questions explored herein offers a means of helping students in coming to know the Fundamental Theorem of Arithmetic as 'theorem in action' a la Vergnaud (Vergnaud, 1982).

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CALCULUS TEACHERS RECOGNITION—IN DIFFERENT CONTEXTS—OF THE CONCEPT OF DERIVATIVE

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Certain difficulties in the teaching of calculus are originated by the present-day organization of the mathematical contents (based on the concept of *limit*) and the way it is communicated. We pose the question if calculus teachers could improve their didactical strategies taking into account the historical evolution of the present-day concept of derivative. The main purpose of this research was to carry out an experimental survey to explore what effects are produced in the conceptualization of differential calculus teachers on this subject have, after they had discussed and studied a textbook especially prepared for it.

The five lessons of the textbook are based on the concepts of *fundamental difference* of discrete variables (the evolution of a given sequence $\{a_n\}$ is studied taking advantage of the substraction $d_n = a_n - a_{n-1}$) and *différence* [L'Hospital 1696] of continuously varying quantities. In the first lesson, we present some results on the connection between the notions of variable and variation in contexts of discrete variation. The second lesson is about the construction of a bridge, based on the graphical representation, between both of the mentioned concepts. In the third lesson we introduce the uses of the concept of dif-ference (handling it algorithmic and conceptually) of continuous variables to determine tangent lines, extrema, inflexion and return points as well as evolutes to curves. The facsimile of the main parts of sections I through V of L'Hospital 1696 that cover the mentioned themes are included in these two lessons, with my Spanish version. In these lessons, the geometrical approach provides visual meaning to analytical results. The fourth lesson treats of the most meaningful elements which conveyed from the manipulation of the differential to that of the derivative; the geometrical interpretation of this concept as a lineal approximation to certain value of any differentiable function is included. Fermat's work on the use of what later on was the concept of derivative is emphasized as well as the work of Lagrange on the algebraization of calculus. Finally, in the fifth lesson, the recognition of the calculus, presented in the previous lessons, is discussed by problems of variation of natural phenomena; namely, the differential equations that represent these phenomena are established.

We discussed the material during four sessions of five hours each one with a first group of calculus teachers who had been taking a program of specialization in the teaching of mathematics for a year before at a university of México. Previously, the teachers had answered the first of two questionnaires which were the main instrument to analyze some results of this survey. These included questions of mathematical contents about different aspects of the calculus which appear in the textbook—discrete variation; graphical, algebraic, and conceptual contexts; physical interpretation; and also on the teaching of some themes—and their interrelations. The two questionnaires are related each other with respect to these aspects. In fact, the second one—answered two weeks after finishing the working sessions—was prepared according to the results noticed in the answers to the first one and to the development of the working sessions.

The teachers failed in relating the algebraic context of the calculus—which they mastered—with other contexts. This is due to the lack of practice in making connections among these and other contexts. Though they could handle a difficult problem in a given context, when a simpler one was posed in another context, they failed to have recourse on the context they were familiar with. The results of the questionaires also show that the teachers could appropriate the calculus of differences to study the short and long run evolution of numerical sequences; they could use the derivative to compute approximate values of a function as part of their conceptualization of the differential calculus; they could distinguish, at least algorithmically, the present-day concept of derivative from the one of *différence*, and it stood out that they were thinking of modifying their teaching activity having options to do it.

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AN INVESTIGATION OF PRESERVICE TEACHERS' UNDERSTANDING OF RATIONAL NUMBERS

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This study analyzed errors concerning rational numbers made by students in a mathematics course for preservice teachers. The analysis attempted to make sense of the reasonings which would lead the students to make these errors.

A guiding principle of the analysis was the belief that some of the persistent misconceptions and errors were sensible from the students' perspective; hence, a goal was to determine that perspective. The students' small-group and whole-class discussions, and their work on quizzes and tests, were qualitatively analyzed to identify trends in their thinking.

A key misconception involved the preferential use of the notion of rational number as "a number which can be expressed as a terminating or repeating decimal," over the notion "a number which can be expressed as the ratio of two integers," with errors resulting in cases of apparent contradiction between the two.

The persistence of this preference, even under inappropriate circumstances and despite instructional presentations designed to connect the concept of "rational number" to the concept of "ratio of two integers," is documented in this study. Also, a model is proposed which takes students' goals into account, in which the students are actually better served by this particular misconception.

This study illustrates the need for further investigation into reconciling students' goals with instructional goals.

Prospective Secondary Mathematics Teachers' Subject Matter Knowledge: The Concept of Limit As An Example

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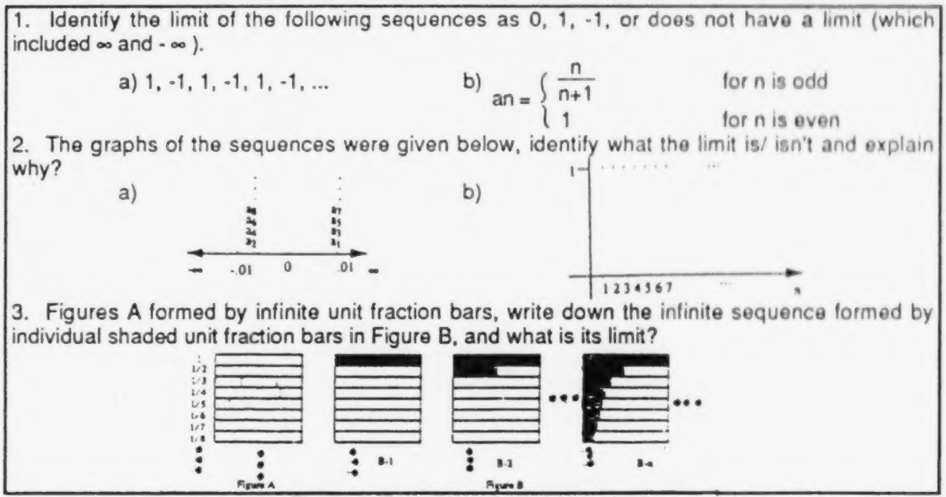
Background: Teachers' subject matter knowledge and its interrelationship with pedagogical content knowledge and curriculum knowledge are still very much unknown. The focus of research on what teachers need to know in order to teach has shifted from quantitatively examining teachers' standardized tests scores to emphasizing knowledge and understanding of facts, concept, and principles and the ways in which they are organized, as well as knowledge about the disciplines. Lee (1992, 1993, 1994) described the inter-relationships among teachers' subject matter knowledge with pedagogical content knowledge and curriculum knowledge; providing a framework for understanding the subject matter knowledge; and the misconceptions possessed by prospective secondary mathematics teachers in terms of the limit concept.

Methods: Thirty-eight prospective secondary mathematics teachers enrolled in mathematics methods course participated in a paper-pencil format survey and four of them were interviewed. The tasks (Figure 1) consisted of five different representations of sequences: listing the first few terms, an algebraic expression, one dimensional graph, two dimensional graph and geometrical figures.



1. Identify the limit of the following sequences as 0, 1, -1, or does not have a limit (which included ∞ and - ∞).

a) 1, -1, 1, -1, 1, -1, ...

b) $an = \begin{cases} \frac{n}{n+1} & \text{for n is odd} \\ 1 & \text{for n is even} \end{cases}$

2. The graphs of the sequences were given below, identify what the limit is/ isn't and explain why?

a)

b)

3. Figures A formed by infinite unit fraction bars, write down the infinite sequence formed by individual shaded unit fraction bars in Figure B, and what is its limit?

Figure 1. Examples of questionnaire items

Results: (1a) Listing the first few terms. Thirteen percent of the subjects thought zero was the limit of [1, -1, 1, -1, 1, -1, ...}, the jumping of 1 and -1 suggested a mental bouncing back and forth movement, eventually stopping at zero. (2a) One dimensional graph. When one-dimensional graph of the sequence {0.01, -0.01, 0.01, -0.01, ...} was given, 5 % of the subjects chose zero as the limit. (1b) Algebraic expression. The limits of rule-oriented sequences turned out to be difficult to identify. (2b)Two dimensional graph. When asked to identify the limit of a constant sequence {1, 1, 1, 1, 1, ...} given by a two-dimensional graph, 5% of the subjects identified zero as the limit. (3) Geometrical figures. Sixteen percent of the subjects could not construct a sequence from this geometrical figures.

Conclusion: Prospective secondary mathematics teachers were able to identify the existence or non-existence of limits for these five different representations of sequences. However, the performance on familiar textbook homework rule-oriented representation is weaker than others except for the geometrical representation. The split-domain sequences caused cognitive conflict for the prospective teachers: they were not able to realize them as unified wholes rather than the combinations of two sub-sequences. Thus, although they were able to compute the limit or verify the non-existence of the limit for different representations of sequences, their ability was restricted to only the straight forward problems.

HANDS-ON ASSESSMENT OF ELEMENTARY TEACHERS' UNDERSTANDING OF WHOLE NUMBER, FRACTION, AND DECIMAL CONCEPTS, PRINCIPLES, AND PROCESSES

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ABSTRACT: The 132-clock hour, year long **MATH FITNESS** project trained second to fifth grade teachers in the use of manipulatives and active involvement teaching strategies to enhance their understanding of mathematical concepts, principles, and processes to improve their teaching of educationally disadvantaged, low achieving Hispanic minority children.

Innovative hands-on assessment items embodied in concrete manipulative materials were developed to test teachers' understanding of whole numbers, fractions, and decimals and their ability to transfer their understanding across multiple embodiments.

Project results support the assertion that teachers can be tested reliably using hands-on and paper and pencil instruments. This combination of instruments added clarity to interpreting cognitive and affective gains and the relationships between them. Notably, substantial and statistically significant cognitive gains were realized without suffering losses in the affective domain as participants generally exhibited reduced anxiety, increased confidence, and improved attitudes.

The **MATH FITNESS** operational plan evolved out of many years of work with teachers of Hispanic minority elementary school children and suggests means for overcoming factors that inhibit maximal mathematics learning and performance of such children. The project's findings support the assertion that teachers can adequately be assessed on their ability to use manipulatives to represent mathematical concepts, principles, and processes and that effective staff development activities can be designed using such information.

PRESERVICE TEACHERS' PERCEPTIONS OF PROCEDURAL AND CONCEPTUAL PROBLEMS

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While problem-solving involves both procedural and conceptual knowledge, the distinction is important for studying knowledge acquisition as well as aspects of student attitudes. Silver (1986) noted that students can be fluent procedurally but not conceptually. Hiebert and Lefevre (1986) discovered that most instruction, from elementary school through college, emphasizes procedural knowledge.

Preservice elementary teachers will eventually be faced with critical decision-making in the classroom. A concern must be their attitudes toward and their beliefs about different kinds of problems they will assign their students.

This study involved the examination of the perceptions of preservice elementary teachers about procedural and conceptual problems. During a second semester mathematics content course, they were asked to respond to questions, on a Likert-type scale, about both types of problems in the area of geometry. The questions involved difficulty, usefulness, confidence, clearness, and original thinking.

The data showed a significant correlation of -.8408 (p=0.036). The conceptual problems were ranked harder but more useful. The procedural problems produced more confidence about the solutions. The students thought that conceptual problems were more confusing and required much more original thinking.

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LOGO IN THE CLASSROOM: RECOGNIZING SOCIO-CULTURAL CONTEXTS IN TECHNOLOGY-BASED LEARNING ENVIRONMENTS

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The goal of our projects is to develop an approach to mathematics learning that recognizes socio-cultural contexts and honors the resulting diversity of mathematical creations by students. In this paper we focus on classroom environments and the mathematical work of children as a framework for this approach.

We have chosen to work with Logo because it is compatible with a constructionist approach allowing students to create projects of their own choosing in an environment that supports the development of mathematical concepts and allows them to pull upon strengths developed in relation to their socio-cultural background. Recognizing and honoring the socio-cultural background of students can be particularly important for children whose home cultures are not that of the dominant culture. Such students often have had experiences which convey messages of failure in mathematics.

We examine an 8-year old girl's efforts to create a rainbow at an African-centered urban school. She first constructed a procedure for making a circle, adapted it to make a semi-circle, and finally modified it to create a series of concentric semicircles. Although Shamia had not worked with circles for nearly a year and had never worked with semi-circles nor varied the size of circles, her determination to create this rainbow led her to explore several mathematical ideas, including the relationship between turn values and circle size, that between turn size and the number of repeats, and the role of decimals for finding numbers between two integers. The character of her work was shaped by her need for ownership of the strategies she used in her rainbow construction which, in turn, meant ownership of the mathematical ideas embedded in her project. Her teacher felt very strongly that the pivotal value of ownership in her engagement with her work was connected to values he saw as integral to her home life and its roots in Cape Verdian culture. This view highlights the school environment where student values and goals are recognized allowing students to create their own message of success in their mathematical activity.

We believe that traditional models of learning (constructivist, socio-historic, etc.) do not adequately value the socio-cultural background of the student and often miss their diverse learning needs. Recognizing the importance of these factors can lead to a robust multi-cultural view of mathematics that supports the development of a positive integration of racial, cultural, and intellectual identity.

The issues we face in developing such an approach involve:

1) How can we validate all of the ways that children construct new knowledge?
2) How can we recognize that some of the ways they construct ideas, emanate from their socio-cultural experience?
3) How can we validate ways of learning that are not reflective of mainstream epistemologies?
4) How do elements (social, cultural, material, pedagogical) of children's environments come together to support their engagement with mathematical ideas?

Our goal is to expand current learning models to be more inclusive of socio-cultural factors, while at the same time recognizing concerns that classroom theory and pedagogy help children learn in ways that facilitate their access to mainstream culture. The challenge is: How we can build on their socio-cultural background in ways that support relationships to new knowledge and provide them with an educational experience that facilitates opportunities for entry into mainstream educational and economic activities.

Teachers' Understanding of Students' Understanding

RESEARCH PAPERS

Delineating The Transformation Of Subject-Matter Knowledge Into Pedagogical Content Knowledge 317
Christine L. Ebert

Pedagogical Content Knowledge, Curricular Knowledge And Teacher Change 324
Beverly S. Rich, Cheryl A. Lubinski, and Albert D. Otto

POSTERS

Integrated Media In Preservice Teacher Education To Explore Children's Mathematical Learning 331
Linda Barron and Elizabeth Goldman

Traditional And Non-Traditional Pre-Service Mathematics: An Investigation Of Factors That Influence The Acquisition Of Pedagogical Content Knowledge 332
Christine L. Ebert and Billie F. Risacher

Pathfinder Analysis: Using Technology To Map The Mathematical Pedagogical Content Knowledge Of Preservice Elementary Teachers 333
Oakley D. Hadfield

DELINEATING THE TRANSFORMATION OF SUBJECT-MATTER KNOWLEDGE INTO PEDAGOGICAL CONTENT KNOWLEDGE

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This study examined the transformation of subject-matter knowledge into pedagogical content knowledge for three prospective secondary mathematics teachers in the content domain of functions and graphs. The prospective secondary teachers, enrolled in a one-semester secondary methods course, were assessed on a wide variety of tasks designed to illuminate the transformation process. The primary objective of this study was to document evidence of the transformation process for three teachers with similar (adequate but not strong) subject-matter knowledge and differing beliefs about the learner and learning mathematics. The findings suggest that the complex endeavor of acquiring pedagogical content knowledge is significantly influenced by the belief structure which prospective teachers hold when they enter the methods class.

Conceptual Framework

The current reform movement in mathematics education suggests that pedagogical content knowledge and teacher subject-matter knowledge are important components of the evolving view of mathematical competence. The emphasis in mathematics as a result of The Curriculum and Evaluation Standards for School Mathematics is being placed on examining various representations of a concept and developing connections between those representations. As a result, classroom teachers (expert, novice, and prospective), who are encouraged to convey to their students the processes in which mathematics is discovered and communicated, experience new demands on their pedagogical content knowledge.

The study of teachers' pedagogical content knowledge in the context of functions and graphing is an important topic within the conceptual frameworks dealing with research on teaching (Brophy, 1991; Shulman, 1986) and research on functions and graphing (Leinhardt, Zaslavsky, & Stein, 1990). Shulman (1986), who identifies teacher subject-matter knowledge as the "missing paradigm" in research on teaching, has inspired much of this work. He describes pedagogical content knowledge as the knowledge "which goes beyond knowledge of subject matter per se to the dimension of subject-matter knowledge for teaching" (Shulman, 1986, p.9).

Though Shulman's description is widely accepted within the research community, the questions concerning where pedagogical content knowledge comes from and what are its origins remain. As preservice teachers prepare for teaching and for making the transition from students to teachers, it is likely that they draw upon several sources of knowledge and beliefs. First, their own subject knowledge will obviously play a significant part in their efforts to design lessons for students. But it is reasonable to suppose that subject-matter knowledge will not completely determine the lessons. Teachers' beliefs about how students learn mathematics, their beliefs about mathematics itself, and their knowledge of teaching, in general, are likely to affect how they design and teach mathematics lessons. Two teachers who share similar subject-matter profiles may teach differently if they differ along these dimensions.

An investigation of prospective secondary teachers' pedagogical content knowledge may be operationalized in the context of a specific content domain. The content domain of functions and graphs is important both from the perspective of being a central topic in the secondary curriculum and also from the perspective of serving as a site in which conceptual understanding is dependent on connecting different representational forms (Leinhardt et al., 1990). The current reform movement in mathematics education also emphasizes the importance of forming connections between these two representations of functions (NCTM Standards, 1989). Thus, the study of pedagogical content knowledge of prospective secondary teachers within the content domain of functions and graphs provides the opportunity to examine the relationship between knowledge and beliefs and instructional practices in an important and central topic in secondary mathematics.

Within the content domain of functions and graphing there have emerged a number of studies which utilize research on students' thinking in this domain to investigate teachers' knowledge about the content and the learner (Even, 1993; Wilson, 1994). This study linked knowledge and beliefs by examining the influence of knowledge in the content area and beliefs about the learner and mathematics on the transformation of subject-matter knowledge into pedagogical content knowledge.

Methodology

The subjects chosen for this study consist of three of the eleven prospective secondary mathematics teachers enrolled in a Secondary Methods course at a major university located in the Mid-Atlantic states. Tasks were designed to assess different aspects of pedagogical content knowledge within the discipline of mathematics and the content area of functions and graphs. The methods class for these prospective teachers was chosen as a research site because it presented the opportunity to examine evidence of the transformation on a variety of assessment tasks. These tasks included 1) a written assessment of subject-matter knowledge of functions and graphs; 2) a beliefs scale that specifically addressed beliefs about the learner, learning mathematics, and mathematics; 3) audiotaped interviews of the teachers' responses to vignettes of students solving problems in the content domain; 4) a unit plan dealing with key topics in the content domain; 5) a video-taped peer-lesson on quadratic functions; and 6) two stimulated-recall interviews of the quadratic functions videotape both before and after reading Standard 6 on functions.

Based on the written assessment of subject-matter knowledge (Task I) three subjects were chosen. These teachers, who represent an intermediate level of expertise with respect to their subject-matter knowledge will henceforth be referred to as Beth, Mark, and Penny. Based on the beliefs scale, a 32-item Likert scale which specifically addressed beliefs about the learner, learning mathematics, and mathematics, a profile of beliefs was generated for each of the subjects. These profiles were then used to determine how beliefs influenced the instructional decisions of the teachers.

Results and Discussion

In the case of Beth we find a prospective teacher who exhibits understanding of students' conceptions of functions in the vignettes task, structures an exploratory quadratic functions lesson that examined revenue, cost, and profit for a pizza shop, and reflects that she chose this kind of lesson because it was "the best way for students to learn - it gets them involved in a problem that involves concepts".

Mark's responses to the vignettes task also indicated that he possessed an insightful understanding of students' conceptions and potential misconceptions of functions. For his video-taped peer-lesson, Mark also chose a lesson that could be characterized as exploratory. He brought in a full-length mirror and placed it lengthwise on the floor, shiny side out so that his standing volunteer had one leg on either side. In his reflection on the lesson, he explained that he brought in the mirror because "it is important to distinguish between linear and quadratic functions...we need to see the difference and the axis of symmetry highlights that difference".

In Penny's case, her responses to the student dilemma's posed in the vignettes task were essentially procedural - "make sure that you plug-in (x+1) for all of the x's in the equation". For the videotaped peer-lesson, Penny chose a lesson that examined various ways of solving quadratic equations - factoring, completing the square, and using the quadratic formula. She began with four clear examples of how to solve these equations and reflected that she liked the lesson because she used "lots of different examples, the students (her peers) were involved, and the order was correct - simple to more complex".

Preliminary results of the beliefs profiles indicate that beliefs about the learners and learning mathematics played an important role in determining the nature of the lessons they designed. On the Beliefs Scale, Beth and Mark were generally in strong agreement with the same statements in which Penny strongly disagreed. Though space will not permit reporting complete profiles for each teacher, two examples of statements along which they differ provides some evidence to support the findings. On the following statements, Beth and Mark strongly agreed and Penny strongly disagreed.

> The role of the teacher in the secondary mathematics classroom is to facilitate learning.
>
> Learning situations should be embedded in authentic problem solving situations that have meaning for the student.

In the vignettes task, the initial opportunity for evidence of pedagogical content knowledge to be revealed, we find that all of the subjects have essentially assumed the role of transmitting knowledge. Beth suggests some attention to specific errors and Mark goes a bit further with the idea of using the error to suggest a misconception that should be examined. However, Penny is the most directive with her response to "make sure that you plug-in (x+1) for all the x's in the equation". On the videotaped lesson, the last instructional task, we begin to see some differences in the three subjects. Both Beth and Mark chose exploratory lessons that utilized student-to-student interactions. Their choice of lesson revealed attention to facilitating learning - Beth because she indicated that applications were the "best way for students to learn concepts" and Mark because he used visualization to enable the students to understand the differences in the symbolic form of equations. Penny continued to assume the role of the teacher who transmits knowledge.

Conclusion

As we consider how the knowledge and beliefs about the learner, mathematics, and pedagogy influence the transformation process, it seems reasonable to suggest that the differences in beliefs contributed to the differences in pedagogical content knowledge revealed in the videotaped lesson. Penny's ambivalence about the use of applications helps to explain her choice of a purely procedural lesson. Her beliefs about the role of the teacher in the secondary mathematics classroom and the way that students learn also supports her choice of lesson. For Beth and Mark, their beliefs would also seem to contribute to the type of lesson they chose. Their belief about the role of the teacher and the importance of authentic problem situations would seem to be essential to support their choice of lesson. To understand the complex process of transforming subject-matter knowledge into pedagogical content knowledge: it is important to investigate the sites where transformed knowledge may be revealed; to examine how knowledge and beliefs about the learner and the content influence the transformation process; and to search for the links between them.

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PEDAGOGICAL CONTENT KNOWLEDGE, CURRICULAR KNOWLEDGE AND TEACHER CHANGE

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This paper introducesan instrument used to help describe teacher change by providing insights into teachers' pedagogical-content and curricular knowledge as well as teachers' beliefs. The instrument consists of 12 open-ended questions dealing with operations, fractions, geometry, number sense, and mathematical reasoning. Questions require teachers to assign a grade level(s) and to discuss what they think their students will do or how they would handle given situations. Responses could be categorized as tearcher directed or student generated. Some responses clearly indicated errors in teachers' understanding of mathematical concepts. The instrument was used as part of the data collection for an on-going project[1] *to prepare novice teachers through collaboration with experienced teachers and univeristy faculty.*

Background

Several instruments currently exist for documenting teacher change within mathematics learning environments. Often the focus of these instruments has been on teachers' attitudes and beliefs or content knowledge. Some have been on teachers' learning styles. Few have explored teachers' pedagogical-content or curricular knowledge. One purpose of our study was to better describe teacher change by devising an instrument that would provide insights into teachers' pedagogical-content and curricular knowledge.

The Importance of Documenting Change in Teachers' Beliefs and Knowledge

Teachers' beliefs about the teaching and learning of mathematics have been shown to play a critical role affecting the learning environment (Thompson, 1992; Brown, Cooney, & Jones, 1990; Floden, McDiarmid, & Weimers, 1990; Greenwood & Parkay, 1989; Hollingsworth, 1989). These beliefs often include the ideas that mathematics is composed of a set of procedural rules to be memorized, that topics should be presented in isolation with no regard for problem solving, and that teaching is telling and showing (Ball & Mosenthal, 1990). Many of these beliefs evolve through teachers' experiences as students, especially during their teacher preparation in college (Duffy & Roehler, 1986; Lampert, 1988). We recognize that these firmly-held beliefs about

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teaching can be obstacles to implementing recommendations in current reform documents which address the way we think about and teach mathematics.

Further, teachers' knowledge about mathematics and about how students learn mathematics affects the decisions they make within the learning environment. However, even in this era of mathematical reform, the content background of inexperienced elementary teachers is frequently poor and they rarely develop a deep understanding of mathematics (Ball & McDiarmid, 1989; Ball, 1990). If teachers are to teach using innovative methods, focusing on understanding, then their own understanding and conceptions of mathematics are of utmost importance (Ball & Mosenthal, 1990). Thus, we believe it is important to assess and develop teachers' pedagogical content and curricular knowledge.

The Project

Considering these findings, we are currently involved in a project that challenges teachers' beliefs and is intended to extend teachers' pedagogical-content and curricular knowledge. With support from NSF, we have undertaken a 5-year project that is designed to prepare better novice teachers, those who base instructional decisions on research findings about children's mathematical thinking. The project incorporates both a preservice and inservice teacher education component since we believe that in order to affect optimal change in the teaching and learning of mathematics, collaboration among various teacher educators is necessary, including those at the university level.

In Spring 1993, 29 experienced K-6 teachers self-selected to participate in a four-week seminar during the following summer. The intent of the seminar was to inform teachers about research on how children think about mathematics. Additionally, in order that they might use their knowledge of students' thinking when making instructional decisions, efforts were made to bring about growth in the teachers' pedagogical-content and curricular knowledge.

Teachers were provided with time each day to read articles. Each week they interviewed students at various levels. The teachers developed interviews to assess students' understandings in the areas of whole number operations, fractions, mathematical reasoning, and place value. In addition, they studied videotapes of K-6 students solving a variety of problems in order that they could see the varied strategies and informal knowledge that students bring to a problem situation. The teachers also solved mathematical problems. During the last week, the teachers planned for the forthcoming school year.

During the past school year, staff and teachers attended bimonthly meetings for two hours after school. Even though these meetings provided time for teachers to share successes and concerns, extending pedagogical-content and curricular knowledge continued to be a major focus. During the entire school year individual staff members were available to visit classrooms and collaborate with teachers about the change process.

This spring 29 preservice teachers were selected to participate in the project. They attended, with the experienced teachers, a two-week summer seminar focusing on problem solving. During the fall semester these preservice teachers are taking a mathematics methods course that has the same focus as the initial four-week seminar for the experienced teachers. The clinical experience of the mathematics methods course will involve the preservice teachers in 1) a teaching experience in the established learning environments and 2) the bi-monthly meetings with the experienced teachers. The cooperative teaching teams and bimonthly meetings will continue to focus on pedagogical-content and curricular knowledge through their during student teaching. A culminating two-week seminar will be held for everyone during the summer.

Data Collection

Baseline data were collected from both the experienced and preservice teachers. This data include a written beliefs survey, a beliefs interview, A Survey on Teaching Mathematics, the Myers-Briggs Profile, taping of a model lesson, and a stimulated-recall interview following the taped lesson. Tapings with a stimulated-recall interview are collected each semester throughout the study for each group. The other measures, except the Myers-Briggs, are collected at the end of each project year.

The focus of the rest of our discussion is on the Survey on Teaching Mathematics instrument. This researcher-designed survey examines pedagogical-content and curricular knowledge, however, beliefs also enter into participants' answers. The survey was piloted in two different forms with 16 questions each. Four questions appeared on both forms. The questions were analyzed as to the types of responses they elicited from the teachers responding. The final survey consists of 12 questions dealing with operations, fractions, geometry, number sense, and mathematical reasoning. These questions focus on how teachers believe their students will respond in given situations, and/or how the teachers themselves would respond to other situations. Teachers are also asked the grade level(s) at which they feel the problems would be appropriate.

Results

The Survey on Teaching Mathematics was given to participating experienced teachers prior to the first four-week seminar. The results of one question from each of the topics operations, fractions, and geometry are given below.

Operations

#1. (From NCTM, 1991) One day your students finish working on addition and subtraction with regrouping. On a written test, many of them "forget" to regroup when they need to in subtraction. Instead they do this:

$$\begin{array}{r} 60 \\ \underline{-28} \\ 48 \end{array}$$

a) What would you do and why?
b) Why is this an appropriate thing to do?

Our experienced teachers indicated that this was primarily a second or third grade problem. These teachers' frequently responded to this situation with one or more of the following statements: 1) review regrouping, practice more computations; 2) check with addition; or 3) use manipulatives. Each response could be categorized as either teacher directed or student generated. For example, "I would review regrouping first, showing each step and using manipulatives..." and "I think I would model this procedure using base-ten blocks. I would also do some work with checking subtraction." versus "I would work one on one with the students and ask them to show me how they got their answer. Many would 'see' their own mistake just by working through it with me." The teachers believed what they were doing was appropriate because "practicing will help students understand the concept of trading" versus "More learning takes place when a person discovers their own mistake and can correct it in their own learning style." One teacher reported that the "lower students" especially needed the teacher to stop and explain if a mistake is made. It was interesting to note that some teachers focused on subtracting down because you can't take 8 away from zero, the larger number always has to be on top they wrote.

Fractions

#4 Suppose you are working with a group of students on addition and subtraction of fractions. Several of them are solving problems as shown below:

$$1/2 + 1/3 = 2/5$$

When asked about their solution, they respond, "Jane made one out of two free throws in the first half and one out of three in the second. So, she made two out of five in the game."

a) What do you respond?
b) Why do you respond in this way?

The experienced teachers believed that this was primarily a fifth grade problem. Responses varied, but it was possible to determine if the teacher understood the mathematics of the problem. For example, "I would respond that their equation is correct if these two 1/2 + 1/3 were ratios. . ." versus "Could be a logical answer the way it's written." Again, responses could be categorized as teacher directed or student generated. "Lead them to discover you have to have a common 'whole'." and "I would write the problem on the board and I would use overhead fraction pieces to show that 1/2 + 1/3 does not equal 2/5." versus "use the same fractions, but talk about pies."

Geometry

#5 (From NCTM, 1991) Suppose you had 64 meters of fence with which you were going to build a pen for your large dog, Bones. What are some different pens you could make if you use all the fencing? What is the pen with the least play space? What is the biggest pen you can make--the one that allows Bones the most play space? Which would be the best for running?

After considering the problems, the students explain that it doesn't matter since all the pens will have the same perimeter--64 meters.

a) Explain why they gave this response.
b) How would you respond to their solution? Explain.

The experienced teachers indicated that this problem could be used at any grade level K-6, however, the most felt it was best for grades 4-6. The teachers responded to the first question in one of three ways. The first indicated that the students were not distinguishing between perimeter and area. The second response indicated that the students were correct and the third reason was that the students did not have enough practice with different shapes and perimeter. Half of the teachers focused on perimeter and did not mention area. Responses ranged from "No matter how the pen is arranged it will total 64 meters. They are looking only at the perimeter distance, not at the possible shapes and area space of these shapes." to "They are correct. No matter how they arrange the fencing, they would have the same perimeter." The second question elicited four different groups of responses: 1) show pictures; 2) revisit the problem; 3) have the students draw possibilities; and 4) use manipulatives (straws, toothpicks, string, sticks, graph paper). The first two responses were teacher directed and 15 of the teachers responded in this way. "I would visually show them 64 meters in the shape of a triangle, rectangle, square, etc." or "I would first of all draw a picture showing the different areas that have a perimeter of 64 meters." Twelve teacher fell into the last two groupings that are student directed. "I would have them draw different pens on graph paper and count the square units to see the difference in play space in different pens." Another stated that "I would respond by giving them sticks to represent the 64 meters at each cooperative grouping and would then challenge them to come up with as many different pens as possible." Only one teacher suggested shapes other than rectangular could be used. The teachers who gave responses in groups one and two tended to place this problem in the upper grades. Those teachers who expressed more student directed ideas tended to give a wider grade level spread for this problem. Teachers felt their responses were appropriate because students needed to visualize or "see" what was going on. The teachers just went about this in different ways.

Discussion

Assessing teachers' change is a primary goal of our project. Our measurement instrument will allow us to document change in teachers' beliefs and knowledge about content and pedagogy. As we continue to question them about their teaching practices, we will be able to discuss how changes in beliefs and knowledge affect the learning environment.

The instrument allows us to compare teacher responses over time, so the development of their understanding of mathematics can be assessed. Also, we can compare the experienced and inexperienced teachers' responses in order to better plan bi-montly meetings, seminars, and other collaborative efforts. We believe our instrument has implications for staff development and preservice teacher education.

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INTEGRATED MEDIA IN PRESERVICE TEACHER EDUCATION TO EXPLORE CHILDREN'S MATHEMATICAL LEARNING

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Over the past six years, researchers at Vanderbilt University, through funding from the National Science Foundation, have studied uses of integrated media to improve the preparation of elementary school teachers. These integrated media applications, or "cases," have been incorporated into mathematics methods courses to provide an authentic context in which preservice teachers can explore issues in teaching and learning mathematics.

A recently developed case of a three-day lesson sequence on geometry, *Investigations in Teaching Geometry*, uses a CD-ROM format to combine QuickTime movies of the fifth-grade class, teacher commentary, children's written products, and related literature references and abstracts. A part of the case, "Growth of Student Learning," contains a series of video clips of children working in pairs on assigned tasks and the children's written reports. Because of the use of three cameras in the videotaping, the prospective teachers may use multiple windows to study how different pairs of children approached the same task or how the same pair approached a sequence of tasks. The case includes other investigations such as the teacher's role and key mathematical ideas as well as technological tools (e.g., a video index for quick access to various segments) that allow prospective teachers to plan and conduct their own investigations.

The poster session will include a demonstration and discussion of components of the case along with a description of the design principles and examples of how the case was used in a mathematics methods course. Informal assessment of the initial field testing of the case will be included, also.

Traditional and Non-Traditional Pre-service Mathematics Teachers: An Investigation of Factors that Influence the Acquisition of Pedagogical Content Knowledge

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This study examined factors that influence the acquisition of pedagogical content knowledge for two groups of prospective secondary mathematics teachers: (1) post-graduate scientists, mathematicians, and engineers employed in research and development enrolled in a part-time mathematics teaching credential program, and (2) undergraduate mathematics majors in their senior year of study leading to a mathematics teaching credential. Both groups, enrolled in a methods of teaching mathematics course, were assessed by a variety of methods to document evidence of their beliefs about learners, learning mathematics, and mathematics, as well as their subject matter knowledge, and how these influenced their instructional practices. The content area was functions and their graphs, recommended as a unifying idea in the high school mathematics curriculum by the NCTM Standards.

The findings suggest the significant influence of beliefs on both groups' instructional practices. Although these two groups were quite different in academic credentials and in actual experience using mathematics, they exhibited quite similar belief profiles. Likewise the instructional practices were similar and suggest definite links to their belief system. This work reveals a consistent pattern of interaction between the initial factors which were considered and instructional practices across pre-service teachers possessing very different educational and professional experiences.

PATHFINDER ANALYSIS: USING TECHNOLOGY TO MAP THE MATHEMATICAL PEDAGOGICAL CONTENT KNOWLEDGE OF PRESERVICE ELEMENTARY TEACHERS

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The purpose of this study was to employ both a computerized conceptual mapping technique and simulated teaching tasks to investigate the development of pedagogical content knowledge of 50 preservice elementary teachers as they completed a course in mathematics teaching methodology. The study utilized a specific computerized data collection technique, know as *Pathfinder Analysis,* that was designed to access the pedagogical knowledge and instructional skills of preservice teachers at a more in-depth level than traditional written examinations. The instrument was designed in the field of psychology where it has been effectively employed in areas such as pilot training and research methods instruction. In this particular study, the format was designed to create relationships between mathematical content, concepts, and manipulatives. Scores on "information", "coherence", and "similarity to instructor" were calculated by the computer, and knowledge structure mappings were produced.

The study also employed the observation and video taping of simulated teaching episodes, in which subjects were given 12 concepts to teach with appropriate mathematics manipulatives. Rating scores on these episodes were then statistically compared to both course exam average and Pathfinder scales. Several significant relationships between variables resulted. The most noteworthy finding was that a significantly greater portion of the variance in simulated teaching success could be explained by pedagogical content knowledge as measured by *Pathfinder Analysis* than could be explained by course examination average. Those students with higher information value and similarity to instructor scores were significantly more successful in the simulated teaching episodes.

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